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# STUDIES ON ROAD TRAFFIC FLOW

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STUDIES  
ON  
ROAD TRAFFIC FLOW

by  
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## Preface

In the past decade, many countries in the world have begun to confront serious problems of air pollution, traffic noise, traffic congestion and traffic accidents which are brought by the rapid growth of the use of motor vehicles. These traffic problems have aroused a great interest of traffic engineers and theoreticians of various kinds in the study of (road) traffic flow. Now this field is called the theory of (road) traffic flow or traffic flow theory. Although a vast amount of scientific investigations have been carried out in the field of the theory of traffic flow since the beginning of this century, many problems still remain open. The reason is that real traffic situations are very complex: they involve human nature on the one hand physical laws of movement on the other.

This thesis is devoted to showing many theoretical results for several basic problems in the theory of traffic flow. The problems discussed in the thesis might be classified into the following three subjects: traffic flow, traffic queue and traffic control. It is to be hoped that the results shown in the thesis will find practical application in the field of traffic flow.



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## Chapter 1. Introduction

### 1. 1. The theory of traffic flow

According to the Statistical Yearbook (1971) published by Statistical Office of the United Nations, the number of motor vehicles in use in the world was 216 million in 1968, of which 170 million were passenger cars, and increased by 71.4 percent between 1960 and 1968. While population in the world increased by 15.9 percent for the same period. In Argentina, the number of motor vehicles in use and population increased by 109 percent and by 13.2 percent, respectively, for the same period. In Australia, the corresponding percentage increases were 52.5 and 17.0; in Brazil, 105 and 26.5; in Canada, 48.3 and 15.9; in France, 95.6 and 9.2; in Fed. Rep. of Germany, 141 and 9.0; in Italy, 265 and 6.2; in the United Kingdom, 80.6 and 5.5; in the United States, 36.5 and 11.3. In particular, Japan showed a remarkable growth in the number of motor vehicles, which increased by 576 percent between 1960 and 1968. While population increased only by 8.4 percent for the same period. According to the Japan Statistical Yearbook (1971), there were 12.8 million motor vehicles registered in 1970. Ratios of motor vehicles to people dropped from one fiftieth in 1960 to

one eighth in 1970. Besides, between 1960 and 1970, the proportion of domestic cargo carried by motor vehicles increased from 76.9 percent to 88.5 percent of the total tonnage of domestic cargo and increased from 15.1 percent to 37.9 percent of the total freight movement, measured in ton-kilometres. For passenger domestic travel during the same period, the proportion of passengers carried by motor vehicles rose from 38.9 percent to 59.0 percent of the total number of passengers carried and from 22.8 percent to 49.4 percent of the total passenger volume, measured in person-kilometres. The statistics mentioned above show that in the post decade there has been a considerable growth in the use of motor vehicles in most countries in the world, particularly, in Japan. The main reason for this fact is that the desire for individual mobility is universal and growing with the rise in standards of living. Therefore the same trend will continue in the immediate future, even though urban areas will undertake extensive programs to improve mass transit. The rapid growth of the use of motor vehicles, however, has brought not only benefits but also serious problems of air pollution, traffic noise, parking difficulties, traffic congestion and safety. For instance, between 1960 and 1970, real lengths of roads per one motor vehicle registered dropped from 513 metres in 1960 to 79



metres in 1970 in Japan. Moreover, in 1969 there were 721 thousand traffic accidents on Japanese roads, which caused 16 thousand persons to be killed, a 35 percent increase over 1960, and 967 thousand persons to be injured, a 234 percent increase over 1960. The trends are the same in most countries and the traffic problems mentioned above become one of the most vital social problems of the world.

The theory of traffic flow is one of the main approaches to the road traffic problems. According to the annotated bibliography compiled by Haight (1964), Crosby wrote a paper entitled "Value of Traffic Census" as early as 1915. This seems to be the earliest scientific research in the field of the theory of traffic flow. In the 1920', however, no remarkable investigations, except a few ones concerning with the statistical analysis of traffic accidents, which followed Crosby's were carried out. In the middle of 1930', Greenshields (1934, 1935) made an experimental study of traffic flow by measuring actual flows and velocities and fitted a straight line to a plot of velocity against concentration. Adams (1936) considered a pedestrian crossing problem for a Poisson traffic flow and obtained a formula for the average wait. Over twenty scientific investigations including Crosby's, Greenshields' and Adams' were

performed by the end of the 1930" and over one thousand by 1964. By the present, probably, over two thousand scientific investigations have been carried out in the field of the theory of traffic flow. In spite of these vast amount of scientific investigations, many problems still remain open. Because real traffic situations are very complex: they involve human nature on the one hand and physical laws of time, space and movement on the other.

Problems which are dealt with in the theory of traffic flow seem to be classified into the following four main subjects: traffic flow, traffic queues, traffic control, and safety and traffic noise. The first subject is concerned with the movement of vehicles on roads and the second is related with queues of vehicles at junctions or bottlenecks. The third subject is concerned with the optimal control of traffic flow through traffic lights and the planning of road network that diminish traffic congestion, air pollution, traffic accidents and so on. The last subject includes the statistical analysis of traffic accidents and the related topic of traffic noise. In the following sections, these main subjects will be surveyed and reviewed in some detail.

## 1. 2. Traffic flow

When a road is empty except for one vehicle, the movement of that vehicle depends principally on the wishes of the driver and is usually of a constant velocity called a desired one. Nearly the same can be said for a traffic flow whose density, or concentration is low. That is, the interactions which occur between vehicles are negligible and each vehicle drives at his own desired velocity. The traffic flow with this property is called a low density one. As the density of traffic flow increases, the interactions between vehicles also increase and queues of vehicles with higher desired velocities form behind slower vehicles. The traffic flow whose density is not low but not so high that no overtaking is possible is called a medium density one. Finally the traffic flow whose density is so high that no overtaking is possible is called a high density one. Thus, the traffic flow is classified into three different types according to its density. On the other hand, several different models, each of which is reasonable for a limited type of three different traffic situations, have been suggested to describe and analyze the traffic flow. These approaches fall largely into the following two categories: deterministic approaches and probabilistic ones.

To begin with, let us be concerned with the

deterministic approaches. One deterministic approach is through kinematic wave theory. This approach is based on fluid dynamic analogies to traffic flow and reasonable for the medium or high density traffic flow. This theory takes no account of properties of individual vehicles. Therefore it can not be expected to explain traffic behaviour in detail but certain results about the propagation of traffic waves and the frequent appearance of shock waves have been obtained. This approach was developed independently by Lighthill and Whitham (1955) and Richards (1956). Generalizations of their theories have been given by De (1956), Greenberg (1959), Bick and Newell (1960), Oliver (1964), Pipes (1965, 1968, 1969) and Munjal and Pipes (1971). Gerlough (1961), Grace and Potts (1964) and Herman and Rothery (1967) have attempted to use kinematic wave theory to analyze the diffusion of traffic platoons.

Another possible approach is to regard the traffic flow as the flow of molecules in a gas and hence it applies to the low or medium density traffic flow. Since this approach is based upon an integro-differential equation of the Boltzmann type, this is called a Boltzmann-like approach. This approach was originated by Prigogine and Andrews (1960) and Prigogine (1961). Generalizations of their theories have been given by



Anderson, Herman and Prigogine (1962), Belescu, Prigogin, Herman and Anderson (1967), Prigogine, Herman and Anderson (1967), Munjal and Pahl (1969), Dutter and Zackor (1971) and Herman and Lam (1971 a, b). Gafarian, Munjal and Pahl (1971), however, have shown that Boltzmann-type integro-differential equations put forward by Prigogine et al give a very poor agreement between experimental data and computed values.

The last of deterministic approaches is through car-following theory. This is concerned with the dynamics of a line of vehicles following one another without overtaking and hence this applies to the high density traffic flow. This approach is based upon a set of differential-difference equations which govern the response of each driver to changes in the speed of the vehicle ahead. Pipes (1953) proposed a differential-difference equation representing a rule for following another vehicle at a safe distance, which was generalized and solved by Chandler, Herman and Montroll (1958), Chow (1958), Kometani and Sasaki (1958), Gazis, Herman and Potts (1959) and Herman, Montroll, Potts and Rothery (1959). Kometani and Sasaki (1959 a, b, 1961 a, b) and Sasaki (1959) deduced a differential-difference equation involving the acceleration of both leading and following

vehicles and discussed the stability of the motion. Many other different differential-difference equations have been suggested by Edie (1961), Gazis, Herman and Rothery (1961), Newell (1961 a, b), Herman and Rothery (1962), Lee (1966), Pipes (1967) and Bexelius (1968). Related works have been done by Kisi (1960), Helly (1961), Herman and Potts (1961), Newell (1962), Gazis, Herman and Weiss (1963), Evans (1966), Unwin and Duckstein (1967), Leipnik (1968), Petigny (1968), Rockwell, Ernst and Hanken (1968), Cosgriff (1969), May and Keller (1969), Weiss (1969 a), Bender and Fenton (1970), Kapur (1971) and Fowkes (1972).

In the above, three deterministic approaches to the traffic flow have been surveyed and reviewed. The traffic flow, however, is not in fact a continuous fluid but is made up of discrete vehicles. Moreover the traffic flow essentially has stochastic properties due to different physical factors such as sight distance, roadside development and grade, different traffic factors such as traffic volume, portion of trucks and parking conditions, and different environmental factors such as driver characteristics, weather, season, speed limits and other traffic controls (see Berry and Belmont (1951)). Hence a stochastic approach seems the most obvious and right line of approaches to the traffic flow. In fact, as

early as 1930', Adams (1936) showed that the times at which vehicles passed a fixed point could be regarded as a Poisson process. Since that time, the Poisson counting distribution and the exponential headway distribution have been substantiated repeatedly by many researchers. Moreover, many new counting or headway distributions have been suggested by Beckmann, McGuire and Winsten (1956), Haight (1958), Oliver (1961, 1962 b), Buckley (1967, 1968), Newell (1967), Dunne, Rothery and Potts (1968), Serfling (1969), Vaughan (1970) and Ashton (1971). Velocity distributions and acceleration noise have been discussed by Berry and Belmont (1951), Wardrop (1952), Montroll (1961), Johns and Potts (1962), Olcott (1965), Helly and Baker (1967) and Mori, Takata and Kisi (1968).

Weiss and Herman (1962) considered the low density traffic flow on an infinitely long road without intersections and showed that when the velocity distribution was continuous, the only counting distribution that remained itself in the long run, was the Poisson distribution, and that the number of vehicles which overtook or were passed by a vehicle moving at a given velocity also was distributed with the Poisson distribution. The first result has been generalized by Breiman (1963), Suzuki (1966) and Thedéen (1967 a, b, 1969 b). Related problems have been considered by Doob

(1953, pp. 404-407), Dobrushin (1956), Kisi (1962), Karlin (1966, page 497), Goldman (1967), Stone (1968) and Brown (1969 b, 1970). The second result and related problems have been discussed by Rényi (1964), Leipnik (1967), Breiman (1969), Brown (1969 a) and Srivastava (1969). Besides, Newell (1955, 1966) discussed the low density traffic flow in which vehicles interacted in such a way as to delay a vehicle when it passed another.

Kometani (1955) considered the low or medium Poisson traffic flow on a two-way two-lane road and obtained the probability that a faster vehicle could overtake a bunch of slow vehicles. In this two-way two-lane model, when the faster vehicle catches up a bunch of slower vehicles, it can overtake immediately the bunch if there is a sufficient gap in the opposing stream of vehicles; if there is not a sufficient gap, it follows the bunch until it finds a sufficient gap. Tanner (1958, 1961) discussed this model and obtained the mean delay and the average velocity of the faster vehicle. Miller (1961 a, b, 1962) dealt with more generalized model in which vehicles traveled in random bunches and overtook at random times and velocities of vehicles were distributed generally, and derived equations for the process of catching-up and overtaking. Related problems have been discussed by Miller (1963 a, 1967, 1970), Gordon and



Newell (1964), Rényi (1964), Erlander (1967, 1971 a, b), Gustavsson (1967), Holland (1967), Oliver and Lam (1967), Warnhuis (1967), Gani and Srivastava (1967), Andrews (1970 a, b), McNeil (1970), Schach (1970), Brill (1971), Morse and Yaffe (1971) and Pahl and Sands (1971).

Finally, the high-density traffic flow in which passing is not allowed has been dealt with by a few researchers. For instance, Newell (1959 b, 1961 a) and Edie and Foote (1961) discussed the traffic flow in tunnels, in which passing was forbidden by law. Hodgson (1968) investigated the time to drive through a no-passing zone.

### 1. 3. Traffic queues

The last of the preceding section has been concerned with the problem of "dynamic delay", which is caused by overtaking in the traffic flow on a two-lane road. This section is concerned with problems of "static delay" (see, Ashton (1966)), which include the problem of the delay incurred by a pedestrian who wants to cross a road, that of the delay incurred by a vehicle wishing to cross or merge into a major traffic stream at an intersection with stop signs, that of the delay of a vehicle at an intersection controlled by a traffic light and so on. The first problem is briefly called the pedestrian crossing problem and the second is called the merging problem. These two problems, however, are essentially the same.

The pedestrian crossing problem was first discussed by Adams (1936) in connection with an attempt to justify the Poisson counting distribution. Tanner (1951) derived the distribution of the delay to pedestrians and distributions of the sizes of groups of pedestrians waiting to cross a road, assuming Poisson arrivals for both vehicles and pedestrians. His results were generalized by Mayne (1954) to the case where interarrival times of vehicles were generally distributed. Related problems have been discussed by Moore (1953), Cohen,

Dearnaley and Hansel (1955), Vuchlic (1967) and Weiss (1967 a). On the other hand, the earliest model of the merging problem, studied originally by Ráff (1951), assumes the merging strategy that a vehicle merges as soon as a gap greater than some predetermined constant, say,  $T$  appears in the major traffic stream. Little (1961), Oliver (1962 a), Oliver and Bisbee (1962), Tanner (1962) and Hawkes (1965, 1966) dealt with this model. This model has been generalized in two different directions. In the first extension, it is assumed that  $T$  is a random variable that varies from vehicle to vehicle, but is fixed for a particular vehicle. This model has been developed by Miller (1961 a), Yeo and Weesakul (1964) and McNeil and Morgan (1968). In the second extension, proposed by Herman and Weiss (1961) and Weiss and Maradudin (1962), it is assumed that  $T$  is a random variable that varies from gap to gap. This model has been generalized by Gaver (1963, 1966), Evans, Herman and Weiss (1964), Weiss (1967 b, 1969 b), Allan (1968), Hawkes (1968) and Wiener and Yagoda (1970). McNeil and Smith (1969) compared the delays of vehicles in the case of the above two models. Reid (1967, 1968) discussed the delay of a right turning vehicle at an uncontrolled intersection. Mine and Mimura (1969) and Blumenfeld and Weiss (1971) investigated the merging problem with an

acceleration lane of finite length. Various aspects of the merging problem have been discussed by Jewell (1964), Ashworth (1968, 1969), Athans (1969), Blumenfeld and Weiss (1970), Breeding (1970) and Gafarian and Walsh (1970).

A large number of researchers have dealt with a queue of vehicles at an intersection controlled by a traffic light, which is briefly called a traffic light queue. A traffic light is called fixed-cycle, if it has a cycle of fixed length. It is called vehicle-actuated, if it contains some type of detectors. In particular, it called semi-vehicle-actuated, if it contains detectors on the only one road of the two intersecting ones. **that** is usually called the minor road. Beckmann, McGuire and Winsten (1956) derived a relation between the mean delay per vehicle and the mean queue-length for a fixed-cycle traffic light queue, assuming constant departure headways and binomial arrivals of vehicles. This model has been discussed by Newell (1960 a), Dunne (1967), Potts (1967) and Smit (1971). Fixed-cycle traffic light queues with constant departure headways and Poisson arrivals have been discussed by Webster (1958), Haight (1959) and Buckley and Wheeler (1964). Fixed-cycle traffic light queues with constant departure headways and stationary and independent arrivals have been



investigated by Uematu (1957), Miller (1963 b), Darroch (1964), Kleinecke (1964), Newell (1956, 1965), McNeil (1968) and Siskind (1970). Vehicle-actuated traffic light queues have been dealt with by Garwood (1940), Darroch, Newell and Morris (1964), Newell (1969) and Newell and Osuna (1969). Semi-vehicle-actuated traffic light queues have been investigated by Haight (1959) and Little (1971). Various problems related to traffic light queues have been discussed by Newell (1959 a), Smeed (1957), Gazis, Herman and Maradudin (1960), Olson and Rothery (1961), Dick (1964), Webster (1964), Anker and Gafarian (1967), Blunden and Pretty (1967), Anker, Gafarian and Gray (1968), Huddart (1969), Thedéen (1969 a) and Dickey and Montgomery (1970).

There are some other traffic situations which incur the static delay. A traffic queue at a roundabout, or traffic circle was discussed by Helly (1964). Traffic jams at bottlenecks which result from flow-stopping incidents such as traffic accidents, mechanical failures and road repairing, have been dealt with by May and Keller (1967), McNeil (1969), Gaver (1969) and Shaw (1970, 1971).

#### 1. 4. Traffic control

Traffic control is the most important subject in the theory of traffic flow and has been discussed by a great many researchers. The most common device for controlling a critical intersection in an urban area is the traffic light. As noted in the preceding section, there are three types of traffic lights in general use: fixed-cycle, vehicle-actuated and semi-vehicle-actuated. Wardrop (1952), Webster (1958), Miller (1963 b), Rangarajan and Oliver (1967) and Allsop (1971 a, b) have discussed optimal signal settings for a fixed-cycle traffic light that minimize the mean delay per vehicle. Dunne and Potts (1964, 1967) proposed some control algorithms for a vehicle-actuated traffic light at an undersaturated intersection which guaranteed that, for any initial state, the traffic light would eventually achieve a limit cycle for which the mean delay per vehicle was minimum. Their control algorithms have been dealt with by Green and Hartley (1966), Grafton and Newell (1967) and Hartley (1969 a, b, c). Related control algorithms have been discussed by Neimark and Fedotokin (1966), Martin-löf (1969) and Fedotokin (1969). Gazis (1964) and Gazis and Potts (1965) obtained conditions for the optimal control of a vehicle-actuated traffic light at an intersection which became oversaturated for some finite length of

time, and extended their results to two intersections. Green (1967) and Gordon (1969) also discussed optimal controls of oversaturated intersections.

When two or more intersections are in close proximity on a main traffic route, some form of linking is necessary to reduce delays and prevent continual stopping. Although there are a few forms of linking, the most effective linked system is the synchronization of traffic lights, which is called sometimes to (flexible) progression system. Newell (1960 b, 1964) discussed delays suffered by vehicles which passed through a sequence of synchronized traffic lights. Morgan and Little (1964) and Little (1966) considered the synchronization of traffic lights that maximized the bandwidth along the main route and formulated a mixed-integer linear program for the problem. The synchronization that minimizes the total delays of vehicles has been dealt with by Bavarez and Newell (1967), Hillier and Rothery (1967) and others. Related problems were discussed by Drew and Pinnel (1965).and Gazis (1965).

The synchronization of traffic lights mentioned above has been applied to road networks in urban areas. Allsop (1968), Stoffer (1968), Teshigawara (1970) and Montgomery, Talavage and Mullen (1972) have dealt with optimal settings of several or more traffic lights in traffic

networks. Longley (1968, 1971) and Ross, Sandys and Schlaefli (1971) have investigated computer control schemes for traffic networks.

Another possible approach to the control of traffic moving over the road network is through the optimal assignment of traffic through the road network. Charnes and Cooper (1958, 1961) formulated linear and nonlinear programs for the problem of the optimal assignment based upon the criterion of minimizing the overall travel time. Whiting and Hillier (1960) suggested an iterative method for finding the shortest route through the road network. Yau (1964) discussed the sensitivity of traffic assignment. The traffic assignment problem under various generalized conditions have been investigated by Almond (1967), Jewell (1967), Mori and Nishimura (1967), Butas (1968), Sakarovitch (1968), Snell et al (1968), Tillman et al (1968), Wollmer (1968), Kirby and Potts (1969), Scott (1969), Halder (1970 a) and Hooi-Tong (1970).

In the above, traffic control systems in urban areas have been discussed. Highway traffic control systems also have been developed and placed in operation in many countries. Various aspects of highway traffic control systems have been discussed by May (1965), Gazis (1967), Miesse (1967), Altman, Pignataro and Yagoda (1968), Rørbech (1968), Gazis and Foote (1969), Yagoda

(1970), Gazia and Knapp (1971) and Black and Gazis (1971). Besides, computer controls of a string of moving vehicles have been investigated by Powner, Anderson and Qualtrough (1969), Anderson and Powner (1970), Peppard and Gourishankar (1970) and Willems (1971).

In the sequel of this section, the planning of traffic network is briefly dealt with. In order to design such a network, the future trip distribution must be forecasted. The trip distribution has been discussed by many researchers including Bieber (1967), Kometani (1967), Heggie (1969), Evans (1970), Ferragu and Sakarovitch (1970), Tomlin (1970), Wagon and Howkind (1970) and Wilson (1970). Prager (1961) discussed the economic design of simple road networks and Tanner (1968) investigated the problem of finding the average travel time for a simple type of motorway layout. Related problems have been dealt with by Ridley (1968), Bergendahl (1969), Dodson (1969), Halder (1970 b) and Thomson (1970).

#### 1. 5. Safety and traffic noise

The subjects of traffic accidents and traffic noise are obviously of great social importance and they are increasingly topics for scientific investigations by researchers in a variety of fields. The two classical papers in the accident field are those of Greenwood and Woods (1919) and of Greenwood and Yule (1920). Greenwood and Woods investigated the frequency of accidents among munition workers, and compared the observed frequency distribution with three theoretical distributions deduced from three different hypotheses. Greenwood and Yule, starting from the Poisson distribution, deduced many more sophisticated models to describe accident phenomena (see, Ashton (1966)). Following their works, a great deal of scientific investigations have been carried out. Recently, Dietz (1967), Treiterer (1967), Lenard (1970) and others have discussed some problems concerned with traffic accidents on a highway.

Although, as noted in the above, the subject of traffic accident has been investigated for fifty long years, it is several years ago that the scientific research on traffic noise was instituted. Johnson and Sounders (1968) presented data on noise generated by freely flowing traffic. Weiss (1970) and Kurze (1971) considered the traffic noise as the noise emitted from

randomly distributed point sources of equal strength on a line and determined the probability distribution and statistical parameters of the intensity of the traffic noise. Waters (1970) discussed the effects of vehicle operating conditions on the noise emitted from the vehicle and Williams (1971) dealt with the influence of road design characteristics on traffic noise. The characteristics of the noise produced by the various major elements including the engine of a vehicle have been investigated by Aspinall (1970), Berry (1970), Lewis (1970), Spellacy (1970), Priede (1971) and others.



## 1. 6: Outline of the thesis

In the preceding sections, the four main subjects in the theory of traffic flow have been surveyed and reviewed. This thesis deals with the following three subjects which are included in the first three main subjects: a low density traffic flow in "traffic flow", traffic light queues in "traffic queues" and optimal signal settings in "traffic control".

Chapters 2 and 3 are concerned with the low density traffic flow explained in Section 1. 2. The main purpose of these chapters is to derive some stochastic properties of the low density traffic flow. In Chapter 2, a time process, a space process and an observation process are introduced as the fundamental stochastic processes associated with the traffic flow. Comparatively speaking, these three random point processes represent observations on the traffic flow at a fixed place, observations from the air at a fixed time and observations by a moving observer, respectively. To begin with, we show that under some conditions, the time process and the space process become inhomogeneous composed Poisson processes. This type of low density inhomogeneous traffic flow are dealt with throughout Chapter 2. It is to be noted that all researchers including Weiss and Herman (1962) have assumed a sort of homogeneity. For this low density

inhomogeneous traffic flow, we will derive various transformations between distributions of time processes, those of space processes and those of observation processes from interpreting velocity as a measure preserving transformation. In Chapter 3, we drop some additional conditions imposed in Chapter 2 and assume only the fundamental characteristic of the low density traffic flow that interactions between vehicles are negligible. That is, Chapter 3 deals with the most general low density traffic flow. To begin with, we define a kind of stochastic integral of the space process at the time origin and that of the time process at the space origin. Then it can be shown that all time, space and observation processes can be expressed in terms of the stochastic integrals. This means that distributions of all processes are completely determined by the distributions of the space process at the time origin and the time process at the space origin. These results will be applied to the inhomogeneous Poisson traffic flow.

Chapters 4, 5 and 6 are concerned with the traffic light queues noted in Section 1. 3. Although a great number of researchers have investigated traffic light queues of various types, their investigations have been limited to simple situations that departure headways and lost times are constant. Chapter 4 deals with a fixed-

cycle and a semi-vehicle-actuated traffic light queues with independent arrivals and independent and identically distributed departure headways and lost times. It can be shown that these traffic light queues are reduced to a generalized model of the GI/G/1 queueing process originated by Lindley (1952). We obtain a necessary and sufficient condition under which the limiting distribution of this generalized model exists. This result leads directly to necessary and sufficient conditions under which those traffic light queues have stationary distributions. Moreover, a successive **approximation** method of the stationary distributions is presented and two typical examples of semi-vehicle-actuated traffic light queues are discussed. Although arrivals are assumed to be independent in Chapter 4, this assumption does not always hold. In Chapter 5, results obtained in Chapter 4 are extended to traffic light queues with dependent arrivals. Explaining in more detail, we will show that the traffic light queues are reduced to a generalized model of Loynes' (1962), and obtain sufficient conditions under which their stationary distributions exist. Chapter 6 deals with fixed-cycle and semi-vehicle -actuated traffic light queues which have departure headways depending upon positions and general arrivals of various types. From the results obtained in Chapters 4 and 5, we can

derive necessary and sufficient conditions or sufficient conditions, according as arrivals are independent or dependent, under which stationary queue length distributions of the traffic light queues exist. A stationary queue length distribution of the fixed-cycle traffic light queue is obtained which has constant departure headways depending upon positions and stationary and independent arrivals. Moreover we will discuss a queue of vehicles in front of a pedestrian crossing which is controlled by a traffic light with detectors of arriving pedestrians.

Chapter 7 is concerned with the traffic light control noted in Section 1. 4. This chapter deals with the optimal signal settings of the fixed-cycle traffic light based upon the criterion of minimum overall delay and that of minimizing the degree of saturation of the whole intersection. As preliminaries, criteria for undersaturation of the whole intersection are derived from the results obtained in Chapter 4 and mean effective green times of streams of lower priorities are determined as linear functions of the green times and the cycle time. It can be shown that the criterion of minimizing the degree of saturation of the whole intersection leads to the well-known rule of thumb for optimal split and that this criterion is closely related

to the criterion of minimum overall delay. Approximation algorithms for the optimal signal settings based upon the latter criterion are presented. These are a refinement of Webster's method and are a combination of linear or mixed-integer programming and one-dimensional minimization technique.

Before proceeding to details, it is to be noted that theorems, lemmas, corollaries and figures are numbered within each chapter and equations within each section. For example, Theorem 1 and Equation (2. 1) represent the first theorem and the first equation of the second section, respectively, in the chapter which they are derived. When they are cited beyond the chapter, that will be stated explicitly. Finally it should be mentioned that the material discussed in Chapter 2 is taken from Mine and Ohno (1968), Chapter 3 from Mine and Ohno (1970 c), Chapter 4 from Mine and Ohno (1971 b), Chapter 5 from Ohno and Mine (1972), Chapter 6 from a submitted paper "Traffic light queues with departure headways depending upon positions" and Chapter 7 also from a submitted paper "Criteria for undersaturation of a whole intersection and optimal signal settings".

## Chapter 2. Low Density Inhomogeneous Composed Poisson Traffic Flow

### 2. 1. Introduction

The purpose of this chapter is to show some statistical properties of a low density inhomogeneous traffic flow. Weiss and Herman (1962) considered a low density Poisson traffic flow on an infinitely long road without intersections or other inhomogeneities and obtained the mean number of vehicles which overtook a vehicle moving at a given velocity and that of vehicles which were passed by the vehicle. Related problems have been discussed by Rényi (1964), Leipnik (1967), Breiman (1969), Brown (1969a) and Srivastava (1969). Besides, the diffusion of traffic platoons has been investigated by Gerlough (1961), Grace and Potts (1964) and Herman and Rothery (1967). The above all researchers assume a sort of homogeneity. As a matter of fact, however, on all roads there exists something that induces a traffic flow to be inhomogeneous, for example, a controlled or uncontrolled intersection, a pedestrian crossing, a bottleneck and so on. Hence it may be important to derive some properties of an inhomogeneous traffic flow.

In this chapter, a traffic flow on a road with an intersection is dealt with. The traffic flow has the

fundamental characteristics that are the existence of time and space and the dispersion of velocities of vehicles (see Haight (1963)). To begin with, a time process and a space process are introduced which represent observations on the traffic flow at a fixed place and observations from the air at a fixed time, respectively. Under some conditions, it can be shown that these processes are inhomogeneous composed Poisson processes. In this sense, the traffic flow dealt with in this chapter is called the low density inhomogeneous composed Poisson traffic flow. It may be noted that the inhomogeneous composed Poisson traffic flow includes as a special case a homogeneous Poisson traffic flow which has been dealt with by many researchers. In Section 2. 3, various transformations between distributions of time processes and those of space processes are derived from interpreting velocity as a measure preserving transformation. These results are concerned with the statistical diffusion of traffic platoons. In Section 2. 4, observation processes are introduced which represent observations on the traffic flow by a moving observer, and their distributions are shown to be determined by distributions of the time process at the space origin and the space process at the time origin. In Section 2. 5, the results obtained in the preceeding sections are applied to a homogeneous Poisson



traffic flow.

## 2. 2. Time processes and space processes.

Consider a low density traffic flow on an  $n$ -lane one-way-road with a junction. It is thought that a traffic flow with relatively low density per lane satisfies the following condition:

- (A): Vehicles travel at their own constant velocities (desired velocities) independently of any other vehicles. For sufficiently small positive number  $h$ , no vehicles have their desired velocities less than  $h$ .

The latter assumption seems rather natural.

Let the space origin be the position of the junction and space interval  $R$  be  $[0, R')$ , where  $R'$  is a positive finite or infinite number. Let time interval  $T$  be  $[0, T']$  and velocity interval  $V$  be  $(0, \infty)$ , where  $T'$  is also a positive finite or infinite number. Let  $(\Omega, \mathcal{B}, \mathcal{P})$  be a probability space;  $\Omega$  is an abstract space of points  $\omega$ ,  $\mathcal{B}$  is a  $\sigma$ -field of subsets of  $\Omega$ , and  $\mathcal{P}$  is a probability measure.

Then, the family of the following stochastic processes

$\{y_r(t, v), r \in R\}$  can be defined on  $(\Omega, \mathcal{B}, \mathcal{P})$ :

$y_r(t, v)$ : the number of vehicles which pass through a point  $r$  during time interval  $(0, t]$  and have their own velocities belonging to velocity interval  $[0, v)$ .

This stochastic process is called a time process at point  $r$  and represents observations on the traffic flow

at point  $r$  (see, Fig. 1). In what follows,  $y_0(t, v)$  and  $y_r(t, \infty)$  are abbreviated to  $y(t, v)$  and  $y_r(t)$ , respectively. In particular,  $y(t, v) (t \in T, v \in V)$  is called the initial time process. Suppose that

(B): For any  $r \in R$ ,  $y_r(t, v)$  is a process with independent increments with respect to both  $t$  and  $v$ .

This means that for arbitrary  $t_i, u_i, v_j$  and  $w_j (i, j=1, 2)$  such that  $0 \leq t_1 < u_1 \leq t_2 < u_2 \leq T$  and  $0 < v_1 < w_1 \leq v_2 < w_2$ ,  $(y_r(u_i, w_j) - y_r(t_i, v_j)) (i, j=1, 2)$  are mutually independent (see for example, Doob (1953), page 96). Moreover, suppose that

(C): For arbitrary  $r \in R$  and  $\varepsilon > 0$ , there exists a positive number  $\delta$  such that for arbitrary  $j=1, 2, \dots$  and

$$0 \leq t_1 < u_1 \leq \dots \leq t_i < u_i \leq \dots \leq t_j < u_j \leq T \text{ which satisfy } \sum_{i=1}^j (u_i - t_i) < \delta,$$

$$\mathbb{P}\{y_r(u_i) - y_r(t_i) = 0 \ (i=1, 2, \dots, j)\} > 1 - \varepsilon.$$

Note that Condition (B) implies that the last inequality is equivalent to

$$\prod_{i=1}^j \mathbb{P}\{y_r(u_i) - y_r(t_i) = 0\} > 1 - \varepsilon.$$

Since by the definition of  $y_r(t, v)$ , for any  $v \in V$  and  $t, u (t < u) \in T$ ,

$y_r(u) - y_r(t) \geq y_r(u, v) - y_r(t, v) \geq 0$  almost surely (a.s.), it is clear that  $y_r(t, v)$  satisfies Condition (C). Renyi (1951) has shown that an integral-valued stochastic process

satisfying Conditions (B) and (C) is an inhomogeneous composed (compound) Poisson process. Denote by  $p(z; y_r(t, v))$  and  $q(z; y_r(t, v))$  the probability generating function of  $y_r(t, v)$  and its natural logarithm, respectively; that is,

$$p(z, \cdot) = \sum_{n=0}^{\infty} P(\cdot = n) z^n$$

and  $q(z, \cdot) = \log p(z, \cdot),$

where  $|z| < 1.$

Since the road has  $n$  lanes, at most  $n$  vehicles arrive simultaneously. Therefore, for  $r \in R$ ,  $t \in T$  and  $v \in V$ ,

$$(2.1) \quad q(z: y_r(t)) = \sum_{k=1}^n \int_0^t \rho_r^{(k)}(u) du (z^k - 1)$$

and

$$(2.2) \quad q(z: y_r(t, v)) = \sum_{k=1}^n \int_0^t \rho_r^{(k)}(u, v) du (z^k - 1)$$

where  $\rho_r^{(k)}(t)$  and  $\rho_r^{(k)}(t, v)$  are nonnegative valued Lebesgue integrable functions and are given by

$$(2.3) \quad \rho_r^{(k)}(t) = \lim_{u \rightarrow t} \frac{P\{y_r(u) - y_r(t) = k\}}{(u - t)}$$

and

$$(2.4) \quad \rho_r^{(k)}(t, v) = \lim_{u \rightarrow t} \frac{P\{y_r(u, v) - y_r(t, v) = k\}}{(u - t)}.$$

Define for  $k=1, 2, \dots, n, r \in R, t \in T$  and  $v \in V, G_r^{(k)}(v, t)$  as

$$(2.5) \quad G_r^{(k)}(v, t) \rho_r^{(k)}(t) = \rho_r^{(k)}(t, v).$$

If  $\mathcal{P}_r^{(k)}(t) > 0$ , then by the definition,  $G_r^{(k)}(v, t)$  is nondecreasing and continuous from the left with respect to  $v$ , and is bounded by zero and unity. If  $\mathcal{P}_r^{(k)}(t) = 0$ , then  $G_r^{(k)}(v, t)$  is defined as a suitable function satisfying the properties mentioned above. Since it can be assumed with little loss of generality that  $G_r^{(k)}(v, t)$  is not singular with respect to Lebesgue measure on  $V$ ,  $G_r^{(k)}(v, t)$  is decomposed as follows (see, Loève (1963), page 178):

$$(2.6) \quad G_r^{(k)}(v, t) = \int_0^v g_r^{(k)}(w, t) dw + \sum_{w_i < v} \bar{g}_r^{(k)}(w_i, t),$$

where  $g_r^{(k)}(v, t)$  is a nonnegative valued Lebesgue integrable function and  $\bar{g}_r^{(k)}(v_i, t)$  is the jump at  $v_i$  of  $G_r^{(k)}(v, t)$ . Note that by Condition (A), jumps of  $G_r^{(k)}(v, t)$  take place at  $v_i$  belonging a countable set  $\{v_i\}$  independent of  $r$ ,  $t$  and  $k$ . Moreover from the definition it follows that  $g_r^{(k)}(v, t) \mathcal{P}_r^{(k)}(t)$  and  $\bar{g}_r^{(k)}(v_i, t) \mathcal{P}_r^{(k)}(t)$  are integrable functions on  $T \times V$ . Consequently, by Equations (2.5) and (2.6) and Fubini's theorem (see Loève (1963), page 136), Equation (2.2) can be written as follows:

$$(2.7) \quad q(z; y_r(t, v)) = \sum_{k=1}^n \left\{ \int_0^v \int_0^t g_r^{(k)}(w, u) \mathcal{P}_r^{(k)}(u) du dw + \sum_{w_i < v} \int_0^t \bar{g}_r^{(k)}(w_i, u) \mathcal{P}_r^{(k)}(u) du \right\} (z^k - 1).$$

For a time interval  $J = (t, u]$  and a velocity interval

$K=[v,w)$ , define a random set function  $y_r(J,K)$  as

$$(2.8) \quad y_r(J,K) = y_r(u,w) + y_r(t,v) - y_r(u,v) - y_r(t,w)$$

Clearly,  $y_r(J,K)$  represents

$y_r(J,K)$ : the number of vehicles which pass through a point  $r$  during time interval  $J$  and have velocities belonging to velocity interval  $K$ .

This random set function also is called the time process at point  $r$  (see, Fig. 1). Throughout this paper, similarly to  $y_r(t,v)$  and  $y_r(t)$ ,  $y_0(J,K)$  and  $y_r(J,V)$  are abbreviated to  $y(J,K)$ , which is called the initial time process, and  $y_r(J)$ , respectively. Since Condition (B) implies that for  $t < u$ ,  $q(z; y_r(u) - y_r(t)) = q(z; y_r(u)) - q(z; y_r(t))$ , it follows from (2.1) and (2.8) that under Conditions (B) and (C),

$$(2.9) \quad q(z; y_r(J)) = \sum_{k=1}^n \int_J \rho_r^{(k)}(t) dt (z^{k-1}).$$

Similarly, from (2.7) and (2.8), under Conditions (B) and (C),

$$(2.10) \quad q(z; y_r(J,K)) = \sum_{k=1}^n \left\{ \int_K \int_J g_r^{(k)}(v,t) \rho_r^{(k)}(t) dt dv + \sum_{v_i \in K} \int_J \bar{g}_r^{(k)}(v_i,t) \rho_r^{(k)}(t) dt \right\} (z^{k-1}).$$

Equations (2.9) and (2.10) mean that for  $m=0,1,2$ ,

...,

$$(2.11) \quad \mathbb{P}\{y_r(J)=m\} = \exp\left\{-\sum_{k=1}^n \int_J f_r^{(k)}(t) dt\right\} \sum_{\ell_1+2\ell_2+\dots+n\ell_n=m} \prod_{k=1}^n \left(\int_J f_r^{(k)}(t) dt\right)^{\ell_k} / \ell_k!$$

and

$$(2.12) \quad \mathbb{P}\{y_r(J,K)=m\} = \exp\left\{-\sum_{k=1}^n \left(\int_K \int_J g_r^{(k)}(v,t) f_r^{(k)}(t) dt dv + \sum_{v_i \in K} \int_J \bar{g}_r^{(k)}(v_i,t) f_r^{(k)}(t) dt\right)\right\} \sum_{\ell_1+2\ell_2+\dots+n\ell_n=m} \prod_{k=1}^n \left(\int_K \int_J g_r^{(k)}(v,t) f_r^{(k)}(t) dt dv + \sum_{v_i \in K} \int_J \bar{g}_r^{(k)}(v_i,t) f_r^{(k)}(t) dt\right)^{\ell_k} / \ell_k!.$$

It should be noted that under the additional assumption

(D): For  $J=(t,u]$ ,  $\lim_{u \rightarrow t} \mathbb{P}\{y_r(J) \geq 1\} / \mathbb{P}\{y_r(J)=1\} = 1$ ,

the time process at point  $r$  reduces to an inhomogeneous Poisson process. That is,

$$(2.13) \quad q(z; y_r(J)) = (z-1) \int_J f_r(t) dt$$

and

$$(2.14) \quad q(z; y_r(J,K)) = (z-1) \left\{ \int_K \int_J g_r(v,t) f_r(t) dt dv + \sum_{v_i \in K} \int_J \bar{g}_r(v_i,t) f_r(t) dt \right\},$$



where  $\rho_r(t)$ ,  $g_r(v,t)$  and  $\bar{g}_r(v,t)$  denote  $\rho_r^{(1)}(t)$ ,  $g_r^{(1)}(v,t)$  and  $\bar{g}_r^{(1)}(v,t)$ , respectively. In fact, Condition (D) implies that for all  $k \geq 2$ ,  $\rho_r^{(k)}(t)$  defined by (2.3) vanish.

In the above, time processes have been discussed. In addition to time processes, the family of the following stochastic processes  $\{x_t(r,v); t \in T\}$  can be defined also on  $(\Omega, \mathcal{B}, P)$ :

$x_t(r,v)$ : the number of vehicles which exist at time  $t$  in space interval  $[0,r)$  and have velocities belonging to velocity interval  $[0,v)$ .

This stochastic process is called a space process at time  $t$  and represents observations on the traffic flow from the air at time  $t$  (see, Fig. 1). Let  $x_0(r,v)$  and  $x_t(r,\infty)$  be abbreviated to  $x(r,v)$  and  $x_t(r)$ , respectively. Suppose that

(B'): For any  $t \in T$ ,  $x_t(r,v)$  is a process with independent increments with respect to both  $r$  and  $v$ .

Moreover, suppose that

(C'): For arbitrary  $t \in T$  and  $\varepsilon > 0$ , there exists a positive number  $\delta$  such that for any  $j=1,2,\dots$  and  $0 \leq r_1 < s_1 \leq \dots \leq r_j < s_j \leq R'$  which satisfy  $\sum_{i=1}^j (s_i - r_i) < \delta$ ,  $P\{x_t(s_i) - x_t(r_i) = 0 \ (i=1,2,\dots,j)\} > 1 - \varepsilon$ .

Then, in the same way as in obtaining Equations (2.1)

and (2. 7), the following equations are derived:

$$(2. 15) \quad q(z; x_t(r)) = \sum_{k=1}^n \int_0^r \lambda_t^{(k)}(s) ds (z^k - 1).$$

and

$$(2. 16) \quad q(z; x_t(r, v)) = \sum_{k=1}^n \left\{ \int_0^v \int_0^r f_t^{(k)}(w, s) \lambda_t^{(k)}(s) ds dw + \sum_{w_i < v} \int_0^r \bar{f}_t^{(k)}(w_i, s) \lambda_t^{(k)}(s) ds \right\} (z^k - 1),$$

where  $\lambda_t^{(k)}(r)$  is defined by

$$(2. 17) \quad \lambda_t^{(k)}(r) = \lim_{s \rightarrow r+0} \frac{P\{x_t(s) - x_t(r) = k\}}{s - r}$$

and  $f_t^{(k)}(v, r)$  and  $\bar{f}_t^{(k)}(v_i, r)$  are Radon-Nykodym's derivative of the absolutely continuous part of  $F_t^{(k)}(v, r)$  defined by the equation analogous to (2. 6) and jumps of its purely discontinuous part.

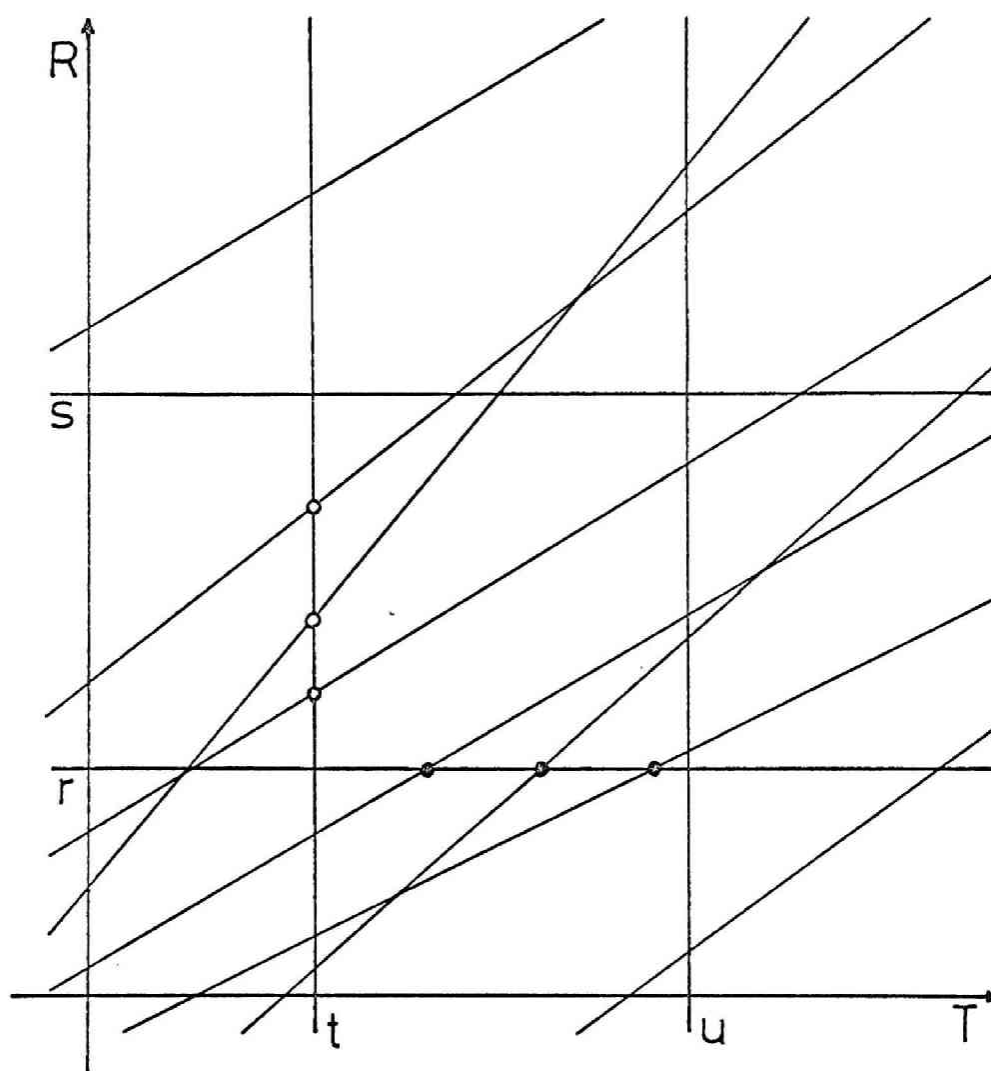
Let  $I=[r, s)$  and define a random set function  $x_t(I, K)$  as

$$(2. 18) \quad x_t(I, K) = x_t(s, w) + x_t(r, v) - x_t(s, v) - x_t(r, w).$$

Clearly,

$x_t(I, K)$ : the number of vehicles which exist at time  $t$  in space interval  $I$  and have velocities belonging to velocity interval  $K$ .

This random set function also is called the space process at time  $t$  (see, Fig. 1). Similarly to  $x(r, v)$  and  $x_t(r)$ ,



$\circ : x_t(I, K)$

$\bullet : y_r(J, K)$

— : trajectories of vehicles

FIG. 1

$x_0(J, K)$  and  $x_t(J, V)$  are abbreviated to  $x(J, K)$ , which is called the initial space process, and  $x_t(J)$ , respectively. In much the same way as in (2. 13) and (2. 14), from Condition (B') and Equations (2. 15), (2. 16) and (2. 18) it follows that

$$(2. 19) \quad q(z; x_t(I)) = \sum_{k=1}^n \int_I \lambda_t^{(k)}(r) dr (z^{k-1})$$

and

$$(2. 20) \quad q(z; x_t(I, K)) = \sum_{k=1}^n \left\{ \int_K \int_I f_t^{(k)}(v, r) \lambda_t^{(k)}(r) dr dv + \sum_{v_i \in K} \int_I \bar{f}_t^{(k)}(v_i, r) \lambda_t^{(k)}(r) dr \right\} (z^{k-1}).$$

Moreover, for  $m=0, 1, 2, \dots$ ,

$$(2. 21) \quad \mathbb{P}\{x_t(I)=m\} = \exp\left\{-\sum_{k=1}^n \int_I \lambda_t^{(k)}(r) dr\right\} \sum_{\ell_1+2\ell_2+\dots+n\ell_n=m} \prod_{k=1}^n \left(\int_I \lambda_t^{(k)}(r) dr\right)^{\ell_k} / \ell_k !$$

and

$$(2. 22) \quad \mathbb{P}\{x_t(I, K)=m\} = \exp\left\{-\sum_{k=1}^n \left(\int_K \int_I f_t^{(k)}(v, r) \lambda_t^{(k)}(r) dr dv + \sum_{v_i \in K} \int_I \bar{f}_t^{(k)}(v_i, r) \lambda_t^{(k)}(r) dr\right)\right\} \sum_{\ell_1+2\ell_2+\dots+n\ell_n=m} \prod_{k=1}^n \left(\int_K \int_I f_t^{(k)}(v, r) \lambda_t^{(k)}(r) dr dv + \sum_{v_i \in K} \int_I \bar{f}_t^{(k)}(v_i, r) \lambda_t^{(k)}(r) dr\right)^{\ell_k} / \ell_k !.$$

It is to be noted that under Condition (D') similar to (D), the space process at time  $t$  becomes an inhomogeneous Poisson process. That is,

$$(2. 23) \quad q(z; x_t(I)) = (z-1) \int_I \lambda_t(r) dr$$

and

$$(2. 24) \quad q(z; x_t(I, K)) = (z-1) \left\{ \int_K \int_I f_t(v, r) \lambda_t(r) dr dv \right. \\ \left. + \sum_{v_i \in K} \int_I \bar{F}_t(v_i, r) \lambda_t(r) dr \right\},$$

where  $\lambda_t(r)$ ,  $f_t(v, r)$  and  $\bar{F}_t(v_i, r)$  denote  $\lambda_t^{(1)}(r)$ ,  $f_t^{(1)}(v, r)$  and  $\bar{F}_t^{(1)}(v_i, r)$ , respectively.

## 2. 3. Transformations between distributions of processes

It has been shown in the preceding section that under Conditions (B) and (C), the time process at an arbitrary point is an inhomogeneous composed Poisson process and under Conditions (B') and (C'), the space process at an arbitrary time is also an inhomogeneous composed Poisson process. The purpose of this section is to find for arbitrary  $t, u \in T$  and  $r, s \in R$ , the transformations from the distributions of  $x_t(I, K)$  and  $y_r(J, K)$  to that of  $x_u(I, K)$  and to that of  $y_s(J, K)$ . For this purpose, it suffices to find the transformations from the distributions of initial processes,  $x(I, K)$  and  $y(J, K)$ , to that of  $x_t(I, K)$  and to that of  $y_r(J, K)$ . To begin with, discuss the case that desired velocities of all vehicles belong to the countable set  $\{v_i\}$ . Then, for all  $t \in T, r \in R$  and  $k=1, 2, \dots$ ,  $F_t^{(k)}(v, r)$  and  $G_r^{(k)}(v, t)$  are purely discontinuous and hence

$$(3.1) \quad F_t^{(k)}(v, r) = \sum_{v_i < v} \bar{F}_t^{(k)}(v_i, r)$$

and

$$(3.2) \quad G_r^{(k)}(v, t) = \sum_{v_i < v} \bar{G}_r^{(k)}(v_i, t).$$

Let  $I=[r, s)$ ,  $J=(t, u]$  and  $K=[v, w)$ . Moreover, let  $K_{1x}=(0, r/t]$ ,  $K_{2x}=(s/t, \infty)$ ,  $K_{1y}=(0, r/u]$  and  $K_{2y}=(r/t, \infty)$ . Denote by

$\bar{x}_t(I, v_i)$  and  $\bar{y}_r(J, v_i)$  the number of vehicles which exist in space interval  $I$  at time  $t$  and have desired velocity  $v_i$  and that of vehicles which pass through point  $r$  in time interval  $J$  and have desired velocity  $v_i$ . That is,

$$(3.3) \quad x_t(I, K) = \sum_{v_i \in K} \bar{x}_t(I, v_i)$$

and

$$(3.4) \quad y_r(J, K) = \sum_{v_i \in K} \bar{y}_r(J, v_i).$$

Condition (A) implies that all vehicles with desired velocity  $v_i \in K_{1x}$  in space interval  $I$  at time  $t$  was in the space interval  $I - tv_i = [r - tv_i, s - tv_i)$  at time zero. Thus, for all  $m = 0, 1, 2, \dots$ ,

$$P\{\bar{x}_t(I, v_i) = m\} = P\{\bar{x}(I - tv_i, v_i) = m\}.$$

This implies that for  $v_i \in K_{1x}$ ,

$$(3.5) \quad p(z; \bar{x}_t(I, v_i)) = p(z; \bar{x}(I - tv_i, v_i)).$$

Similarly, for  $v_i \in K_{2x}$ ,

$$(3.6) \quad p(z; \bar{x}_t(I, v_i)) = p(z; \bar{y}((t-s/v_i, t-r/v_i], v_i)),$$

for  $v_i \in K_{1y}$ ,

$$(3.7) \quad p(z; \bar{y}_r(J, v_i)) = p(z; \bar{x}([r - uv_i, r - tv_i), v_i))$$

and for  $v_i \in K_{2y}$ ,

$$(3.8) \quad p(z; \bar{y}_r(J, v_i)) = p(z; \bar{y}(J - r/v_i, v_i)).$$



Equations (3. 5) through (3. 8) lead to the following lemma.

Lemma 1.

Under Conditions (A) through (C), (B') and (C'), suppose that for all  $r \in R$ ,  $t \in T$  and  $k=1, 2, \dots, n$ ,  $F_t^{(k)}(v, r)$  and  $G_r^{(k)}(v, t)$  are purely discontinuous. Then, for  $I=[r, s) \subset R$ ,  $J=(t, u] \subset T$  and  $K=[v, w) \subset V$ ,

$$q(z; x_t(I, K \cap K_{1x})) = \sum_{k=1}^n \left\{ \sum_{v_i \in K \cap K_{1x}} \int_I \bar{F}^{(k)}(v_i, r - tv_i) \lambda^{(k)}(r - tv_i) dr \right\} (z^k - 1),$$

$$q(z; x_t(I, K \cap K_{2x})) = \sum_{k=1}^n \left\{ \sum_{v_i \in K \cap K_{2x}} \int_I \bar{G}^{(k)}(v_i, t - r/v_i) \rho^{(k)}(t - r/v_i) / v_i dr \right\} (z^k - 1),$$

$$q(z; y_r(J, K \cap K_{1y})) = \sum_{k=1}^n \left\{ \sum_{v_i \in K \cap K_{1y}} \int_J v_i \bar{F}^{(k)}(v_i, r - tv_i) \lambda^{(k)}(r - tv_i) dt \right\} (z^k - 1)$$

and

$$q(z; y_r(J, K \cap K_{2y})) = \sum_{k=1}^n \left\{ \sum_{v_i \in K \cap K_{2y}} \int_J \bar{G}^{(k)}(v_i, t - r/v_i) \rho^{(k)}(t - r/v_i) dt \right\} (z^k - 1),$$

where  $K_{1x} = (0, r/t]$ ,  $K_{2x} = (s/t, \infty)$ ,  $K_{1y} = (0, r/u]$  and  $K_{2y} = (r/t, \infty)$ .

Proof

Let us prove the first equation. Condition (B') and (3. 3)

imply that

$$q(z; x_t(I, K \cap K_{1x})) = \sum_{v_i \in K \cap K_{1x}} q(z; \bar{x}_t(I, v_i)).$$

Therefore, by (3. 5),

$$q(z; x_t(I, K \cap K_{1x})) = \sum_{v_i \in K \cap K_{1x}} q(z; \bar{x}(I - tv_i, v_i)).$$

Thus, combination of (2. 20) and (3. 1) for the initial space process leads to

$$\begin{aligned} q(z; x_t(I, K \cap K_{1x})) &= \sum_{k=1}^n \left\{ \sum_{v_i \in K \cap K_{1x}} \int_{I - tv_i} \bar{f}^{(k)}(v_i, r) \lambda^{(k)}(r) dr \right\} (z^k - 1) \\ &= \sum_{k=1}^n \left\{ \sum_{v_i \in K \cap K_{1x}} \int_I \bar{f}^{(k)}(v_i, r - tv_i) \lambda^{(k)}(r - tv_i) dr \right\} (z^k - 1). \end{aligned}$$

In a similar way, the other equations can be proved by (2. 14), (2. 20) and (3. 1) through (3. 8). The proof is concluded.

Let  $K_{3x} = (K_{1x} \cup K_{2x})^c = (r/t, s/t]$  and  $K_{3y} = (K_{1y} \cup K_{2y})^c = (r/u, r/t]$ , where  $K^c$  means the complement of  $K$  on  $V$ . It is legitimate to assume that the initial time process and the initial space process are mutually independent. Then, in the same way as in (3. 5) through (3. 8), for  $v_i \in K_{3x}$ ,

$$(3. 9) \quad p(z; \bar{x}_t(I, v_i)) = p(z; \bar{x}([0, s - tv_i], v_i)) p(z; \bar{y}((0, t - r/v_i], v_i))$$

and for  $v_i \in K_{3Y}$ ,

$$(3.10) \quad p(z; \bar{y}_r(J, v_i)) = p(z; \bar{x}([0, r - tv_i], v_i)) p(z; \bar{y}((0, u - r/v_i], v_i)).$$

The following lemma results from Equations (3.9) and (3.10).

Lemma 2.

Under the same conditions as in Lemma 1, for  $I = [r, s] \subset R$ ,  $J = (t, u] \subset T$  and  $K = [v, w) \subset V$ ,

$$q(z; x_t(I, K \cap K_{3X})) = \sum_{k=1}^n \left\{ \sum_{v_i \in K \cap K_{3X}} \left[ \int_{tv_i}^s \bar{f}^{(k)}(v_i, r - tv_i) \lambda^{(k)}(r - tv_i) dr + \int_r^{tv_i} \bar{g}^{(k)}(v_i, t - r/v_i) \rho^{(k)}(t - r/v_i) / v_i dr \right] (z^k - 1) \right\}$$

and

$$q(z; y_r(J, K \cap K_{3Y})) = \sum_{k=1}^n \left\{ \sum_{v_i \in K \cap K_{3Y}} \left[ \int_t^{r/v_i} \bar{f}^{(k)}(v_i, r - tv_i) \lambda^{(k)}(r - tv_i) dt + \int_{r/v_i}^u \bar{g}^{(k)}(v_i, t - r/v_i) \rho^{(k)}(t - r/v_i) dt \right] (z^k - 1) \right\},$$

where  $K_{3X} = (r/t, s/t]$  and  $K_{3Y} = (r/u, r/t]$ .

Proof.

As in the proof of Lemma 1, by Condition (B'), (3.3) and (3.9),

$$q(z; x_t(I, K \cap K_{3x})) = \sum_{v_i \in K \cap K_{3x}} \left\{ q(z; \bar{x}([0, s - tv_i], v_i)) + q(z; \bar{y}((0, t - r/v_i], v_i)) \right\}.$$

Consequently, combination of (2. 14), (2. 20), (3. 1) and (3. 2) leads to the first equation to be proved. The second equation can be proved in much the same way. The proof is concluded.

It is clear that

$$(3. 11) \quad x_t(I, K) = \sum_{j=1}^3 x_t(I, K \cap K_{jx}) \text{ a.s.},$$

and

$$(3. 12) \quad y_r(J, K) = \sum_{j=1}^3 y_r(J, K \cap K_{jy}) \text{ a.s.}$$

Hence, by Conditions (B) and (B'), Lemmas 1 and 2 lead directly to the following theorem.

### Theorem 1.

Under Conditions (A) through (C), (B') and (C'), suppose that for all  $r \in R$ ,  $t \in T$  and  $k=1, 2, \dots, n$ ,  $F_t^{(k)}(v, r)$  and  $G_r^{(k)}(v, t)$  are purely discontinuous. Then for  $I=[r, s] \subset R$ ,  $J=(t, U] \subset T$  and  $K=[v, w] \subset V$ ,

$$\begin{aligned} q(z; x_t(I, K)) = & \sum_{k=1}^n (z^k - 1) \left\{ \sum_{v_i \in K \cap K_{1x}} \int_I \bar{f}^{(k)}(v_i, r - tv_i) \lambda^{(k)}(r - tv_i) dr + \sum_{v_i \in K \cap K_{2x}} \int_I \bar{g}^{(k)}(v_i, t - r/v_i) \rho^{(k)}(t - r/v_i) / \right. \\ & \left. v_i dr + \sum_{v_i \in K \cap K_{3x}} \left[ \int_{tv_i}^s \bar{f}^{(k)}(v_i, r - tv_i) \lambda^{(k)}(r - tv_i) dr \right. \right. \end{aligned}$$

$$+ \int_r^{tv_i} \bar{g}^{(k)}(v_i, t-r/v_i) f^{(k)}(t-r/v_i)/v_i dr] \}$$

and

$$\begin{aligned} q(z; y_r(J, K)) = & \sum_{k=1}^n (z^{k-1}) \left\{ \sum_{v_i \in K \cap K_{1y}} \int_J v_i \bar{f}^{(k)}(v_i, r-tv_i) \lambda^{(k)}(r \right. \\ & - tv_i) dt + \sum_{v_i \in K \cap K_{2y}} \int_J \bar{g}^{(k)}(v_i, t-r/v_i) f^{(k)}(t \\ & - r/v_i) dt + \sum_{v_i \in K \cap K_{3y}} \left[ \int_t^{r/v_i} v_i \bar{f}^{(k)}(v_i, r-tv_i) \lambda^{(k)}(r \right. \\ & \left. - tv_i) dt + \int_{r/v_i}^u \bar{g}^{(k)}(v_i, t-r/v_i) f^{(k)}(t-r/v_i) dt \right] \}, \end{aligned}$$

where  $K_{1x} = (0, r/t]$ ,  $K_{2x} = (s/t, \infty)$ ,  $K_{3x} = (r/t, s/t]$ ,  $K_{1y} = (0, r/u]$ ,  $K_{2y} = (r/t, \infty)$  and  $K_{3y} = (r/u, r/t]$ .

This theorem shows that for arbitrary  $r \in R$  and  $t \in T$ , distributions of  $x_t(I, K)$  and  $y_r(J, K)$  are completely constructed by distributions of the initial time and space processes.

Next, consider the case that  $F_t^{(k)}(v, r)$  and  $G_r^{(k)}(v, t)$  are absolutely continuous with respect to Lebesgue measure on  $V$ . In this case,

$$(3.13) \quad F_t^{(k)}(v, r) = \int_0^v f_t^{(k)}(v, r) dv$$

and

$$(3.14) \quad G_r^{(k)}(v, t) = \int_0^v g_r^{(k)}(v, t) dv.$$

Suppose that

(E): For arbitrary  $t \in T$ ,  $k=1,2,\dots,n$  and almost every-  
where (a.e.)  $r \in R$ ,  $f_t^{(k)}(v,r)$  is a.e. continuous in  
 $v \in V$  and for arbitrary  $r \in R$ ,  $k=1,2,\dots,n$  and a.e.  
 $t \in T$ ,  $g_r^{(k)}(v,t)$  is continuous in a.e.  $v \in V$ .

Then, the following theorem similar to Theorem 1 holds.

Theorem 2.

Under Conditions (A) through (C), (B') and (C'), suppose  
that for all  $r \in R$ ,  $t \in T$  and  $k=1,2,\dots,n$ ,  $F_t^{(k)}(v,r)$  and  
 $G_r^{(k)}(v,t)$  are absolutely continuous with respect to  
Lebesgue measure on  $V$  and satisfy Condition (E).

Then, for  $I=[r,s] \subset R$ ,  $J=[t,u] \subset T$  and  $K=[v,w] \subset V$ ,

$$\begin{aligned} q(z; x_t(I,K)) = & \sum_{k=1}^n (z^k - 1) \left\{ \int_I \left[ \int_{K \cap K_{1x}} f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv) dv \right. \right. \\ & + \int_{K \cap K_{2x}} g^{(k)}(v, t-r/v) \rho^{(k)}(t-r/v) / v dv \Big] dr \\ & + \int_{K \cap K_{3x}} \left[ \int_{tv}^s f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv) dr \right. \\ & \left. \left. + \int_r^{tv} g^{(k)}(v, t-r/v) \rho^{(k)}(t-r/v) / v dr \right] dv \right\} \end{aligned}$$

and

$$\begin{aligned} q(z; y_r(J,K)) = & \sum_{k=1}^n (z^k - 1) \left\{ \int_J \left[ \int_{K \cap K_{1y}} v f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv) dv \right. \right. \\ & \left. \left. + \int_{K \cap K_{2y}} g^{(k)}(v, t-r/v) \rho^{(k)}(t-r/v) dv \right] dt \right\} \end{aligned}$$

$$+ \int_{K \cap K_{3y}} \left[ \int_t^{r/v} v f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv) dt + \int_{r/v}^u g^{(k)}(v, t-r/v) \rho^{(k)}(t-r/v) dt \right] dv \Big\},$$

where  $K_{1x}=(0, r/t]$ ,  $K_{2x}=(s/t, \infty)$ ,  $K_{3x}=(r/t, s/t]$ ,  $K_{1y}=(0, r/u]$ ,  $K_{2y}=(r/t, \infty)$  and  $K_{3y}=(r/u, r/t]$ .

Proof.

Let us prove the first equation. From Equations (2. 20) and (3. 13),

$$q(z: x_t(I, K)) = \sum_{k=1}^n \left\{ \int_K \int_I f_t^{(k)}(v, r) \lambda_t^{(k)}(r) dr dv \right\} (z^k - 1).$$

Since under Condition (E),  $\int_I f_t^{(k)}(v, r) \lambda_t^{(k)}(r) dr$  ( $k=1, 2, \dots, n$ ) are continuous in a.e.  $v$ , they are integrable in the sense of Riemann. Let  $\{K_{mi}=(v_{mi}, v_{mi+1}]; i=0, \dots, m-1\}$  be an properly chosen division of  $K$  such that  $v=v_{m0} < v_{m1} < \dots < v_{mi_1} = r/t < v_{mi+1} < \dots < v_{mi_2} = s/t < v_{mi_2+1} < \dots < v_{mm} = w$  and  $\max_i \{v_{mi+1} - v_{mi}\} \rightarrow 0$ , as  $m \rightarrow \infty$ . Then by the definition of Riemann integral,  $q(z: x_t(I, K)) = \lim_{m \rightarrow \infty} \sum_i \left\{ \sum_{k=1}^n (z^k - 1) (v_{mi+1} - v_{mi}) \int_I f_t^{(k)}(v_{mi}, r) \lambda_t^{(k)}(r) dr \right\}$ . On the other hand, (3. 5) implies that for arbitrary  $v_{mi} (i < i_1)$ ,

$$\begin{aligned} & \sum_{k=1}^n (z^k - 1) (v_{mi+1} - v_{mi}) \int_I f_t^{(k)}(v_{mi}, r) \lambda_t^{(k)}(r) dr \\ &= \sum_{k=1}^n (z^k - 1) (v_{mi+1} - v_{mi}) \int_{I - tv_{mi}} f^{(k)}(v_{mi}, r) \lambda^{(k)}(r) dr, \end{aligned}$$

(3. 6) implies that for arbitrary  $v_{mi} (i \geq i_2)$ ,

$$\begin{aligned} & \sum_{k=1}^n (z^k - 1) (v_{mi+1} - v_{mi}) \int_I f_t^{(k)}(v_{mi}, r) \lambda_t^{(k)}(r) dr \\ &= \sum_{k=1}^n (z^k - 1) (v_{mi+1} - v_{mi}) \int_{t-s/v_{mi}}^{t-r/v_{mi}} g^{(k)}(v_{mi}, r) \rho^{(k)}(r) dr \end{aligned}$$

and (3. 9) implies that for arbitrary  $v_{mi} (i_1 \leq i < i_2)$ ,

$$\begin{aligned} & \sum_{k=1}^n (z^k - 1) (v_{mi+1} - v_{mi}) \int_I f_t^{(k)}(v_{mi}, r) \lambda_t^{(k)}(r) dr \\ &= \sum_{k=1}^n (z^k - 1) (v_{mi+1} - v_{mi}) \left\{ \int_0^{s-tv_{mi}} f^{(k)}(v_{mi}, r) \lambda^{(k)}(r) dr \right. \\ & \quad \left. + \int_0^{t-r/v_{mi}} g^{(k)}(v_{mi}, r) \rho^{(k)}(r) dr \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} q(z; x_t(I, K)) &= \sum_{k=1}^n (z^k - 1) \left\{ \int_{K \cap K_{1x}} \int_{I-tv} f^{(k)}(v, r) \lambda^{(k)}(r) dr \right. \\ & \quad + \int_{K \cap K_{2x}} \int_{t-s/v}^{t-r/v} g^{(k)}(v, r) \rho^{(k)}(r) dr \\ & \quad + \int_{K \cap K_{3x}} \left[ \int_0^{s-tv} f^{(k)}(v, r) \lambda^{(k)}(r) dr \right. \\ & \quad \left. \left. + \int_0^{t-r/v} g^{(k)}(v, r) \rho^{(k)}(r) dr \right] \right\}. \end{aligned}$$

Thus, the first equation can be obtained by change of variables. The second equation also can be derived from the same way as in the proof of the first equation. The proof is concluded.

Combination of Theorems 1 and 2 leads immediately to the



following theorem.

Theorem 3.

Suppose that Conditions (A) through (C), (B'), (C') and (E) are satisfied. Then for  $I=[r,s] \subset \mathbb{R}$ ,  $J=(t,u] \subset \mathbb{T}$  and  $K=(v,w] \subset V$ ,

$$\begin{aligned} q(z; x_t(I, K)) = & \sum_{k=1}^n (z^k - 1) \left\{ \int_I \left[ \int_{K \cap K_{1x}} f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv) dv \right. \right. \\ & + \sum_{v_i \in K \cap K_{1x}} \bar{f}^{(k)}(v_i, r-tv_i) \lambda^{(k)}(r-tv_i) \left. \right] dr \\ & + \int_I \left[ \int_{K \cap K_{2x}} g^{(k)}(v, t-r/v) \rho^{(k)}(t-r/v) / v dv \right. \\ & + \sum_{v_i \in K \cap K_{2x}} \bar{g}^{(k)}(v_i, t-r/v_i) \rho^{(k)}(t-r/v_i) / v_i \left. \right] dr \\ & + \int_{K \cap K_{3x}} \left[ \int_{tv}^s f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv) dr \right. \\ & + \int_r^{tv} g^{(k)}(v, t-r/v) \rho^{(k)}(t-r/v) / v dr \left. \right] dv \\ & + \sum_{v_i \in K \cap K_{3x}} \left[ \int_{tv_i}^s \bar{f}^{(k)}(v_i, r-tv_i) \lambda^{(k)}(r-tv_i) dr \right. \\ & + \int_r^{tv_i} \bar{g}^{(k)}(v_i, t-r/v_i) \rho^{(k)}(t-r/v_i) / v_i \left. \right] dr \left. \right\} \end{aligned}$$

and

$$\begin{aligned} q(z; y_r(J, K)) = & \sum_{k=1}^n (z^k - 1) \left\{ \int_J \left[ \int_{K \cap K_{1y}} v f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv) dv \right. \right. \\ & + \sum_{v_i \in K \cap K_{1y}} v_i \bar{f}^{(k)}(v_i, r-tv_i) \lambda^{(k)}(r-tv_i) \left. \right] dt \end{aligned}$$

$$\begin{aligned}
& + \int_J \left[ \int_{K \cap K_{2y}} g^{(k)}(v, t-r/v) f^{(k)}(t-r/v) dv \right. \\
& \quad + \sum_{v_i \in K \cap K_{2y}} \bar{g}^{(k)}(v_i, t-r/v_i) f^{(k)}(t-r/v_i) \left. \right] dt \\
& + \int_{K \cap K_{3y}} \left[ \int_t^{r/v} v f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv) dt \right. \\
& \quad + \int_{r/v}^u g^{(k)}(v, t-r/v) f^{(k)}(t-r/v) dt \left. \right] dv \\
& + \sum_{v_i \in K \cap K_{3y}} \left[ \int_t^{r/v_i} v_i \bar{f}^{(k)}(v_i, r-tv_i) \lambda^{(k)}(r-tv_i) dt \right. \\
& \quad + \int_{r/v_i}^u \bar{g}^{(k)}(v_i, t-r/v_i) f^{(k)}(t-r/v_i) dt \left. \right] \Big\},
\end{aligned}$$

where  $K_{1x}=(0, r/t]$ ,  $K_{2x}=(s/t, \infty)$ ,  $K_{3x}=(r/t, s/t]$ ,  $K_{1y}=(0, r/u]$ ,  $K_{2y}=(r/t, \infty)$  and  $K_{3y}=(r/u, r/t]$ .

This theorem shows that distributions of inhomogeneous composed Poisson process  $x_t(I, K)$  and  $y_r(J, K)$  are completely determined by the distributions of initial processes  $x(I, K)$  and  $y(J, K)$ . Comparison between Theorem 3 and Equations (2. 10) and (2. 20) yields the following corollary:

Corollary.

Suppose that the same conditions as in Theorem 3 are satisfied. Then for arbitrary  $t \in T$ ,  $k=1, 2, \dots, n$  and a.e.  $r \in R$ ,

$$f_t^{(k)}(v, r) \lambda_t^{(k)}(r) = f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv), \quad \text{a.e. } v \in K_{1x}$$

and  $f_t^{(k)}(v, r) \lambda_t^{(k)}(r) = g^{(k)}(v, t-r/v) f^{(k)}(t-r/v)/v$ , a.e.  $v \in K_{1x}^c$ ;

for arbitrary  $r \in R$ ,  $k=1, 2, \dots, n$  and a.e.  $t \in T$ ,

$$g_r^{(k)}(v, t) f_r^{(k)}(t) = v f^{(k)}(v, r-tv) \lambda^{(k)}(r-tv), \text{ a.e. } v \in K_{1x}$$

and  $g_r^{(k)}(v, t) f_r^{(k)}(t) = g^{(k)}(v, t-r/v) f^{(k)}(t-r/v)$ , a.e.  $v \in K_{1x}^c$ ,

where  $K_{1x} = (0, r/t]$ .

Suppose that Conditions (D) and (D') are satisfied. Then, since (2. 10) and (2. 20) reduce to (2. 14) and (2. 24), respectively, Theorem 3 reduces to the following theorem.

Theorem 4.

Suppose that Conditions (A) through (E) and (B') through (D') are satisfied. Then for  $I=[r, s) \subset R$ ,  $J=(t, u] \subset T$  and  $K=(v, w] \subset V$ ,

$$\begin{aligned} q(z; x_t(I, K)) = & (z-1) \left\{ \int_I \left[ \int_{K \cap K_{1x}} f(v, r-tv) \lambda(r-tv) dv \right. \right. \\ & + \sum_{v_i \in K \cap K_{1x}} \bar{f}(v_i, r-tv_i) \lambda(r-tv_i) \left. \right] dr \\ & + \int_I \left[ \int_{K \cap K_{2x}} g(v, t-r/v) f(t-r/v) / v dv \right. \\ & + \sum_{v_i \in K \cap K_{2x}} \bar{g}(v_i, t-r/v_i) f(t-r/v_i) / v_i \left. \right] dr \\ & + \int_{K \cap K_{3x}} \left[ \int_{tv}^s f(v, r-tv) \lambda(r-tv) dr \right. \\ & + \left. \int_r^{tv} g(v, t-r/v) f(t-r/v) / v dr \right] dv \end{aligned}$$

$$+ \sum_{v_i \in K \cap K_{3x}} \left[ \int_{tv_i}^s \bar{f}(v_i, r-tv_i) \lambda(r-tv_i) dr + \int_r^{tv_i} \bar{g}(v_i, t-r/v_i) f(t-r/v_i) / v_i dr \right] \Big\}$$

and

$$\begin{aligned} q(z; y_r(J, K)) = (z-1) \Big\{ & \int_J \left[ \int_{K \cap K_{1y}} v f(v, r-tv) \lambda(r-tv) dv + \sum_{v_i \in K \cap K_{1y}} v_i \bar{f}(v_i, r-tv_i) \lambda(r-tv_i) \right] dt \\ & + \int_J \left[ \int_{K \cap K_{2y}} g(v, t-r/v) f(t-r/v) dv + \sum_{v_i \in K \cap K_{2y}} \bar{g}(v_i, t-r/v_i) f(t-r/v_i) \right] dt \\ & + \int_{K \cap K_{3y}} \left[ \int_t^{r/v} v f(v, r-tv) \lambda(r-tv) dt + \int_{r/v}^u g(v, t-r/v) f(t-r/v) dt \right] dv \\ & + \sum_{v_i \in K \cap K_{3y}} \left[ \int_t^{r/v_i} v_i \bar{f}(v_i, r-tv_i) \lambda(r-tv_i) dt + \int_{r/v_i}^u \bar{g}(v_i, t-r/v_i) f(t-r/v_i) dt \right] \Big\}, \end{aligned}$$

where  $K_{1x} = (0, r/t]$ ,  $K_{2x} = (s/t, \infty)$ ,  $K_{3x} = (r/t, s/t]$ ,  $K_{1y} = (0, r/u]$ ,  $K_{2y} = (r/t, \infty)$  and  $K_{3y} = (r/u, r/t]$ .

This theorem shows that distributions of inhomogeneous Poisson processes  $x_t(I, K)$  and  $y_r(J, K)$  are completely

determined by the distributions of initial processes.

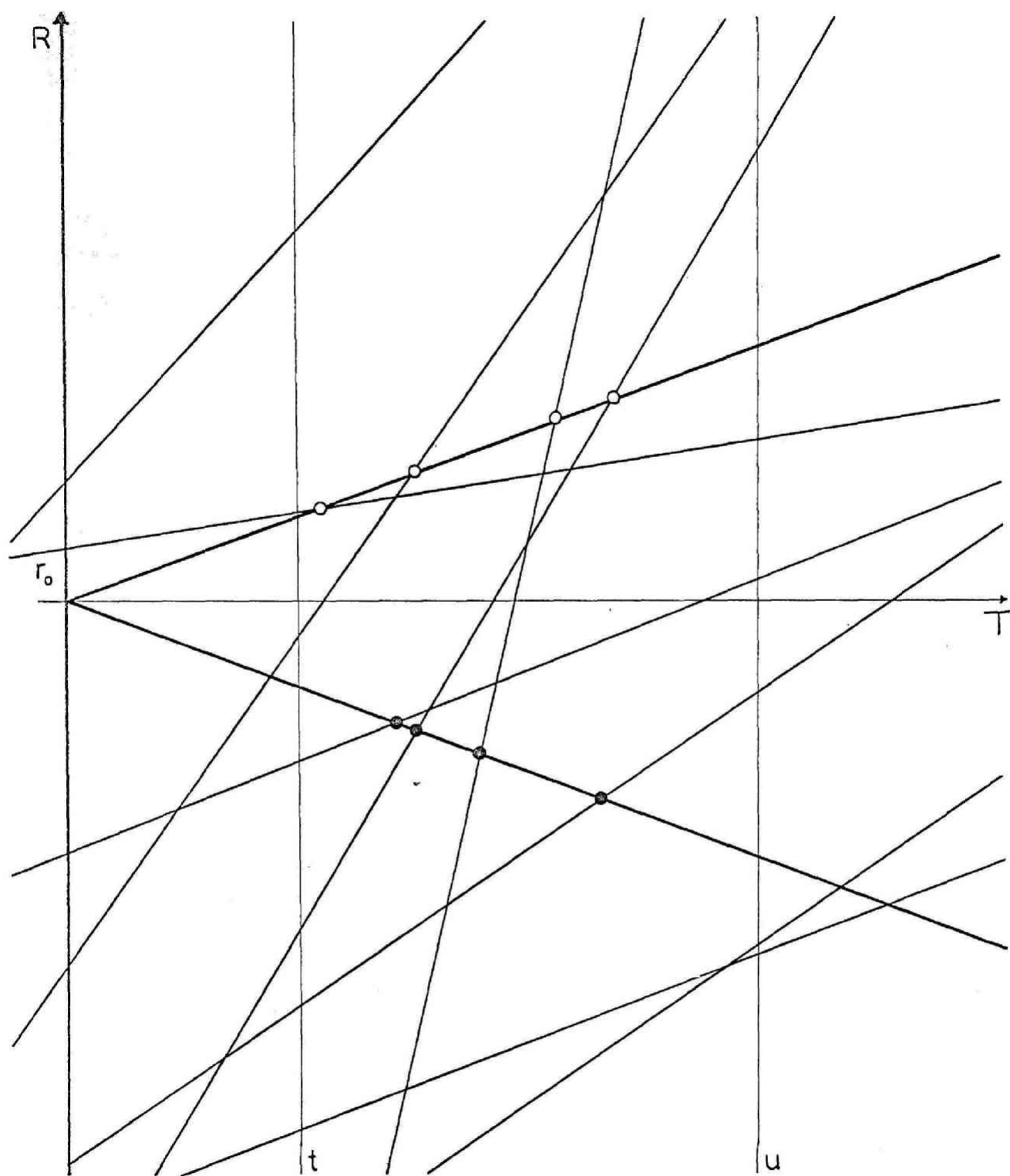
#### 2. 4. Observation process by a moving observer

Let us deal with observations on the traffic flow by a moving observer with constant velocity  $v^*$  (for example, an observer on a car or a helicopter). Define observation processes  $X^-(J, K; v^*, r_0)$  and  $X^+(J, K; v^*, r_0)$  on  $(\Omega, \mathcal{B}, \mathcal{P})$  as follows (see, Fig. 2):

$X^-(J, K; v^*, r_0)$ : the number of those vehicles with their own velocities belonging to  $K$  which are observed during a time interval  $J$  by an observer who starts from a point  $r_0$  at time 0 and is moving with a velocity  $v^*$  in the opposite direction to the traffic flow.

$X^+(J, K; v^*, r_0)$ : the number of those vehicles with their own velocities belonging to  $K$  which are observed during a time interval  $J$  by an observer who starts from a point  $r_0$  at time 0 and is moving with a velocity  $v^*$  in the same direction to the traffic flow.

Note that a vehicle is recorded when the observer catches up with it or is overtaken by it. The purpose of this section is to reveal the differences between distributions of  $x_t(I, K)$  and  $y_r(J, K)$  and ones of  $X^-(J, K; v^*, r_0)$  and



$\circ : X^+(J, K; v^*, r_0), \quad \bullet : X^-(J, K; v^*, r_0).$   
 — : trajectories of vehicles.

FIG. 2

$X^+(J, K; v^*, r_0)$ . For this purpose, it suffices to find expressions of distributions of  $X^-(J, K; v^*, r_0)$  and  $X^+(J, K; v^*, r_0)$  in terms of ones of initial processes.

Now define the following three processes:

$z(I, J, K)$ : the number of vehicles which exist or existed in a space interval  $I$  during a time interval  $J$  and have their own velocities belonging to a velocity interval  $K$ .

$x_{tu}(I, K)$ : the number of those vehicles with their own velocities belonging to a velocity interval  $K$  which exist in a space interval  $I$  not only at time  $t$  but also at time  $u (t < u)$ .

$y_{rs}(J, K)$ : the number of those vehicles with their own velocities belonging to a velocity interval  $K$  which pass through not only a point  $r$  but also a point  $s (r < s)$  during a time interval  $J$ .

Let  $J = (t, u]$  and  $K \subset V$ . Then,

$$(4.1) \quad X^-(J, K; v^*, r_0) = z(I, J, K) \text{ a.s.,}$$

where  $I = [r_0 - v^*u, r_0 - v^*t)$ ,

and

$$(4.2) \quad X^+(J, K; v^*, r_0) = x_{tu}(I, K) + y_{rs}(J, K) \text{ a.s.,}$$

where  $r = r_0 + v^*t$ ,  $s = r_0 + v^*u$  and  $I = [r, s)$ .

To begin with, let us discuss the three processes



defined above. Since for  $I=[r,s)$  and  $J=(t,u]$ ,

$$z(I,J,K)=x_t(I,K)+y_r(J,K) \text{ a.s.,}$$

the following lemma is derived from Theorem 3.

Lemma 3.

Suppose that the same conditions as in Theorem 3 are satisfied. Then for  $I=[r,s) \subset R$ ,  $J=(t,u] \subset T$  and  $K=(v,w) \subset V$ ,

$$\begin{aligned} q(z: z(I,J,K)) = & \sum_{k=1}^n (z^k - 1) \left\{ \int_{K \cap K_{1y}} \int_{r=uv}^{s-tv} f^{(k)}(v,r) \lambda^{(k)}(r) dr dv \right. \\ & + \sum_{v_i \in K \cap K_{1y}} \int_{r=uv_i}^{s-tv_i} f^{(k)}(v_i,r) \lambda^{(k)}(r) dr \\ & + \int_{K \cap K_{2x}} \int_{t=s/v}^{u-r/v} g^{(k)}(v,t) \rho^{(k)}(t) dt dv \\ & + \sum_{v_i \in K \cap K_{2x}} \int_{t=s/v_i}^{u-r/v_i} g^{(k)}(v_i,t) \rho^{(k)}(t) dt \\ & + \int_{K \cap K_{1y}^c \cap K_{2x}^c} \int_0^{s-tv} f^{(k)}(v,r) \lambda^{(k)}(r) dr dv \\ & + \sum_{v_i \in K \cap K_{1y}^c \cap K_{2x}^c} \int_0^{s-tv_i} f^{(k)}(v_i,r) \lambda^{(k)}(r) dr \\ & + \int_{K \cap K_{1y}^c \cap K_{2x}^c} \int_0^{u-r/v} g^{(k)}(v,t) \rho^{(k)}(t) dt dv \\ & \left. + \sum_{v_i \in K \cap K_{1y}^c \cap K_{2x}^c} \int_0^{u-r/v_i} g^{(k)}(v_i,t) \rho^{(k)}(t) dt \right\}, \end{aligned}$$

where  $K_{1y}=(0, r/u]$  and  $K_{2x}=(s/t, \infty)$ .

Denote by  $\bar{x}_{tu}(I, v_i)$  and  $\bar{y}_{rs}(I, v_i)$  the number of vehicles with velocity  $v_i$  which exist in a space interval  $I$  not only at time  $t$  but also at time  $u$  ( $>t$ ) and that of vehicles with velocity  $v_i$  which pass through not only a point  $r$  but also a point  $s$  ( $>r$ ) during a time interval  $J$ . Let  $K_1=(0, s/u]$  and  $K_2=(0, |I|/|J|]$ , where  $|\cdot|$  denotes the Lebesgue measure of  $\cdot$ . Then, in a similar way to (3. 5) through (3. 10), for  $v_i \in K_{1x} \cap K_2$ ,

$$(4. 3) \quad p(z; \bar{x}_{tu}(I, v_i)) = p(z; \bar{x}([r-tv_i, s-uv_i], v_i)),$$

for  $v_i \in K_{1x}^c \cap K_1 \cap K_2$ ,

$$(4. 4) \quad p(z; \bar{x}_{tu}(I, v_i)) = p(z; \bar{x}([0, s-uv_i], v_i)) p(z; \bar{y}((0, t-r/v_i], v_i)),$$

for  $v_i \in K_1^c \cap K_2$ ,

$$(4. 5) \quad p(z; \bar{x}_{tu}(I, v_i)) = p(z; \bar{y}((u-s/v_i, t-r/v_i], v_i)),$$

for  $v_i \in K_1 \cap K_2^c$ ,

$$(4. 6) \quad p(z; \bar{y}_{rs}(J, v_i)) = p(z; \bar{x}([s-uv_i, r-tv_i], v_i)),$$

for  $v_i \in K_{1x} \cap K_1^c \cap K_2^c$ ,

$$(4. 7) \quad p(z; \bar{y}_{rs}(J, v_i)) = p(z; \bar{x}([0, r-tv_i], v_i)) p(z; \bar{y}((0, u-s/v_i], v_i))$$

and for  $v_i \in K_{1x}^c \cap K_2^c$ ,

$$(4.8) \quad p(z; \bar{y}_{rs}(J, v_i)) = p(z; \bar{y}((t-r/v_i, u-s/v_i], v_i)).$$

Equations (4.3) through (4.8) lead to the following lemma in the same way as in Theorem 3. To avoid duplication, the proof is omitted.

Lemma 4.

Suppose that the same conditions as in Theorem 3 are satisfied. Then, for  $I=[r,s] \subset \mathbb{R}$ ,  $J=(t,u] \subset \mathbb{T}$  and  $K=(v,w] \subset V$ ,

$$\begin{aligned} q(z; x_{tu}(I, K)) = & \sum_{k=1}^n (z^{k-1}) \left\{ \int_{K \cap K_{1x} \cap K_2} \int_{r-tv}^{s-uv} f^{(k)}(v, r) \lambda^{(k)}(r) dr dv \right. \\ & + \sum_{v_i \in K \cap K_{1x} \cap K_2} \int_{r-tv_i}^{s-uv_i} i_{\bar{f}}^{(k)}(v_i, r) \lambda^{(k)}(r) dr \\ & + \int_{K \cap K_{1x}^c \cap K_2} \int_{t-r/v}^{u-s/v} g^{(k)}(v, r) \rho^{(k)}(r) dr dv \\ & + \sum_{v_i \in K \cap K_{1x}^c \cap K_2} \int_{t-r/v_i}^{u-s/v_i} i_{\bar{g}}^{(k)}(v_i, r) \rho^{(k)}(r) dr \\ & + \int_{K \cap K_{1x}^c \cap K_1 \cap K_2} \int_0^{s-uv} f^{(k)}(v, r) \lambda^{(k)}(r) dr dv \\ & + \sum_{v_i \in K \cap K_{1x}^c \cap K_1 \cap K_2} \int_0^{s-uv_i} i_{\bar{f}}^{(k)}(v_i, r) \lambda^{(k)}(r) dr \\ & + \int_{K \cap K_{1x}^c \cap K_1 \cap K_2} \int_0^{t-r/v} g^{(k)}(v, r) \rho^{(k)}(r) dr dv \\ & \left. + \sum_{v_i \in K \cap K_{1x}^c \cap K_1 \cap K_2} \int_0^{t-r/v_i} i_{\bar{g}}^{(k)}(v_i, r) \rho^{(k)}(r) dr \right\} \end{aligned}$$

and

$$\begin{aligned}
q(z; y_{rs}(J, K)) = & \sum_{k=1}^n (z^{k-1}) \left\{ \int_{K \cap K_1 \cap K_2^c} \int_{s-uv}^{r-tv} f^{(k)}(v, t) \lambda^{(k)}(t) dt dv \right. \\
& + \sum_{v_i \in K \cap K_1 \cap K_2^c} \int_{s-uv_i}^{r-tv_i} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt \\
& + \int_{K \cap K_{1x}^c \cap K_2^c} \int_{t-r/v}^{u-s/v} g^{(k)}(v, t) \rho^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1x}^c \cap K_2^c} \int_{t-r/v_i}^{u-s/v_i} \bar{g}^{(k)}(v_i, t) \rho^{(k)}(t) dt \\
& + \int_{K \cap K_{1x} \cap K_1^c \cap K_2^c} \int_0^{r-tv} f^{(k)}(v, t) \lambda^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1x} \cap K_1^c \cap K_2^c} \int_0^{r-tv_i} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt \\
& + \int_{K \cap K_{1x} \cap K_1^c \cap K_2^c} \int_0^{u-s/v} g^{(k)}(v, t) \rho^{(k)}(t) dt dv \\
& \left. + \sum_{v_i \in K \cap K_{1x} \cap K_1^c \cap K_2^c} \int_0^{u-s/v_i} \bar{g}^{(k)}(v_i, t) \rho^{(k)}(t) dt \right\},
\end{aligned}$$

where  $K_{1x}(0, r/t]$ ,  $K_1(0, s/u]$  and  $K_2 = (0, |I|/|J|]$ .

Combination of (4. 1) and Lemma 3 and that of (4. 2) and Lemma 4 lead directly to the following theorem:

Theorem 5.

Suppose that Conditions (A) through (C), (B'), (C') and (E) are satisfied. Then for  $I=[r, s] \subset R$ ,  $J=(t, u] \subset T$  and  $K=(v, w] \subset V$ ,

$$\begin{aligned}
q(z; X^-(J, K; v^*, r_0)) = & \sum_{k=1}^n (z^{k-1}) \left\{ \int_{K \cap K_{1y}} \int_{r_0 - (v+v^*)u}^{r_0 - (v+v^*)t} f^{(k)}(v, \right. \\
& \left. t) \lambda^{(k)}(t) dt dv \right. \\
& + \sum_{v_i \in K \cap K_{1y}} \int_{r_0 - (v_i+v^*)u}^{r_0 - (v_i+v^*)t} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt \\
& + \int_{K \cap K_{2x}} \int_{t(1+v^*/v) - r_0/v}^{u(1+v^*/v) - r_0/v} g^{(k)}(v, t) \rho^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{2x}} \int_{t(1+v^*/v_i) - r_0/v_i}^{u(1+v^*/v_i) - r_0/v_i} \bar{g}^{(k)}(v_i, t) \rho^{(k)}(t) dt \\
& + \int_{K \cap K_{1y}^c \cap K_{2x}^c} \int_0^{r_0 - (v+v^*)t} f^{(k)}(v, t) \lambda^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1y}^c \cap K_{2x}^c} \int_0^{r_0 - (v_i+v^*)t} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt \\
& + \int_{K \cap K_{1y}^c \cap K_{2x}^c} \int_0^{u(1+v^*/v) - r_0/v} g^{(k)}(v, t) \rho^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1y}^c \cap K_{2x}^c} \int_0^{u(1+v^*/v_i) - r_0/v_i} \bar{g}^{(k)}(v_i, \\
& \left. t) \rho^{(k)}(t) dt \right\}
\end{aligned}$$

and

$$\begin{aligned}
q(z; X^+(J, K; v^*, r_0)) = & \sum_{k=1}^n (z^{k-1}) \left\{ \int_{K \cap K_{1x} \cap K^*} \int_{r_0 + (v^*-v)t}^{r_0 + (v^*-v)u} f^{(k)}(v, \right. \\
& \left. t) \lambda^{(k)}(t) dt dv \right. \\
& + \sum_{v_i \in K \cap K_{1x} \cap K^*} \int_{r_0 + (v^*-v_i)t}^{r_0 + (v^*-v_i)u} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt
\end{aligned}$$

$$\begin{aligned}
& + \int_{K \cap K_1 \cap K^*} \int_{u(1-v^*/v)-r_0/v}^{t(1-v^*/v)-r_0/v} g^{(k)}(v, t) f^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_1 \cap K^*} \int_{u(1-v^*/v_i)-r_0/v_i}^{t(1-v^*/v_i)-r_0/v_i} g^{(k)}(v_i, t) f^{(k)}(t) dt dv_i \\
& + \int_{K \cap K_{1x}^c \cap K_1^c \cap K^*} \int_0^{r_0+(v^*-v)u} f^{(k)}(v, t) \lambda^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1x}^c \cap K_1^c \cap K^*} \int_0^{r_0+(v^*-v_i)u} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt \\
& + \int_{K \cap K_{1x}^c \cap K_1^c \cap K^*} \int_0^{t(1-v^*/v)-r_0/v} g^{(k)}(v, t) f^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1x}^c \cap K_1^c \cap K^*} \int_0^{t(1-v^*/v_i)-r_0/v_i} g^{(k)}(v_i, t) f^{(k)}(t) dt \\
& + \int_{K \cap K_1^c \cap (K^*)^c} \int_{r_0-(v-v^*)u}^{r_0-(v-v^*)t} f^{(k)}(v, t) \lambda^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_1^c \cap (K^*)^c} \int_{r_0-(v_i-v^*)u}^{r_0-(v_i-v^*)t} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt \\
& + \int_{K \cap K_{1x}^c \cap (K^*)^c} \int_{t(1-v^*/v)-r_0/v}^{u(1-v^*/v)-r_0/v} g^{(k)}(v, t) f^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1x}^c \cap (K^*)^c} \int_{t(1-v^*/v_i)-r_0/v_i}^{u(1-v^*/v_i)-r_0/v_i} g^{(k)}(v_i, t) f^{(k)}(t) dt \\
& + \int_{K \cap K_{1x} \cap K_1 \cap (K^*)^c} \int_0^{r_0-(v-v^*)t} f^{(k)}(v, t) \lambda^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1x} \cap K_1 \cap (K^*)^c} \int_0^{r_0-(v_i-v^*)t} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt
\end{aligned}$$

$$\begin{aligned}
& + \int_{K \cap K_{1x} \cap K_1 \cap (K^*)^c} \int_0^{u(1-v^*/v) - r_0/v} g^{(k)}(v, t) f^{(k)}(t) dt dv \\
& + \sum_{v_i \in K \cap K_{1x} \cap K_1 \cap (K^*)^c} \int_0^{u(1-v^*/v_i) - r_0/v_i} \bar{g}^{(k)}(v_i, t) \\
& \quad f^{(k)}(t) dt \},
\end{aligned}$$

where  $K_{1x} = (0, v^* + r_0/t]$ ,  $K_1 = (v^* + r_0/u, \infty)$  and  $K^* = (0, v^*]$ .

This theorem shows that the distributions of observation processes  $X^-(J, K; v^*, r_0)$  and  $X^+(J, K; v^*, r_0)$  are determined completely by the distributions of initial processes. In case of  $r_0 = 0$ , the second equation in Theorem 5 reduces to

$$\begin{aligned}
(4.9) \quad q(z; X^+(J, K; v^*, 0)) = & \sum_{k=1}^n (z^k - 1) \left\{ \int_{K \cap K^*} \int_{(v^*-v)t}^{(v^*-v)u} f^{(k)}(v, t) \lambda^{(k)}(t) dt dv \right. \\
& + \sum_{v_i \in K \cap K^*} \int_{(v^*-v_i)t}^{(v^*-v_i)u} \bar{f}^{(k)}(v_i, t) \lambda^{(k)}(t) dt \\
& + \int_{K \cap (K^*)^c} \int_{t(1-v^*/v)}^{u(1-v^*/v)} g^{(k)}(v, t) f^{(k)}(t) dt dv \\
& \left. + \sum_{v_i \in K \cap (K^*)^c} \int_{t(1-v^*/v_i)}^{u(1-v^*/v_i)} \bar{g}^{(k)}(v_i, t) f^{(k)}(t) dt \right\}.
\end{aligned}$$

Suppose that, in addition to conditions in Theorem 5, Conditions (D) and (D') are satisfied. Then the initial processes become inhomogeneous Poisson processes and the distributions of  $X^-(J, K; v^*, r_0)$  and  $X^+(J, K; v^*, r_0)$  are given by Theorem 5 with  $\lambda^{(1)}$ ,  $f^{(1)}$ ,  $\bar{f}^{(1)}$ ,  $\bar{g}^{(1)}$ ,  $g^{(1)}$

and  $\bar{g}^{(1)}$  replaced by  $\lambda, \mathfrak{f}, f, \bar{f}, g$  and  $\bar{g}$ , respectively,  
 and  $\lambda^{(k)} = \mathfrak{f}^{(k)} = 0$  ( $k \geq 2$ ).



## 2. 5. Application to homogeneous Poisson traffic flow

In this section, let us apply the results in the preceding sections to a simple case that time processes and space processes are homogeneous Poisson processes. In order to avoid formal complexity, throughout this section suppose that  $G_r(v, t)$  and  $F_t(v, r)$  are absolutely continuous with respect to Lebesgue measure on  $V$ . Therefore, the distributions of time processes and space processes can be written as:

$$(5. 1) \quad q(z; x_t(I, K)) = (z-1)\lambda |I| \int_K f(v) dv.$$

and

$$(5. 2) \quad q(z; y_r(J, K)) = (z-1)\rho |J| \int_K g(v) dv.$$

Hence it follows from Corollary that for a.e.  $v \in V$ ,

$$(5. 3) \quad \lambda f(v) = \rho g(v)/v.$$

This relation implies that

$$(5. 4) \quad \lambda = \rho \int_V g(v)/v dv$$

and

$$(5. 5) \quad \int_V v f(v) dv = 1 / \int_V g(v)/v dv.$$

Equations (5. 3) through (5. 5) are familiar in traffic flow theory (see, Haight (1963), page 163). Moreover,

it follows from (5. 3) and Theorem 5 that

$$(5. 6) \quad q(z; X^-(J, K; v^*, r_0)) = (z-1)\lambda |J| \int_K (v+v^*) f(v) dv \\ = (z-1)\rho |J| \int_K (1+v^*/v) g(v) dv$$

and

$$(5. 7) \quad q(z; X^+(J, K; v^*, r_0)) = (z-1)\lambda |J| \left\{ \int_{K \cap K^*} (v^*-v) f(v) dv \right. \\ \left. + \int_{K \cap (K^*)^c} (v-v^*) f(v) dv \right\} \\ = (z-1)\rho |J| \left\{ \int_{K \cap K^*} (v^*/v-1) g(v) dv \right. \\ \left. + \int_{K \cap (K^*)^c} (1-v^*/v) g(v) dv \right\}.$$

Equation (5. 7) agrees with the result obtained by Weiss and Herman (1962) and Rényi (1964).

## 2. 6. Concluding remarks

In this chapter it is shown that random set functions  $y_r(J,K)$  and  $x_t(I,K)$  can be constructed by usual point processes  $y_r(t,v)$  and  $x_t(r,v)$  and that the distributions of  $y_r(J,K)$ ,  $x_t(I,K)$ ,  $X^-(J,K; v^*, r_0)$  and  $X^+(J,K; v^*, r_0)$  are completely determined in terms of the distributions of  $y(J,K)$  and  $x(I,K)$ . Therefore, once the distributions of initial processes are given or known, the distributions of all processes are completely determined.

Throughout this chapter, Conditions (A) through (C), (B') and (C') are assumed. However, it is open to question whether all low density traffic flows satisfy Conditions (B), (C), (B') and (C'). These conditions will be dropped in the next chapter.

## Chapter 3. Low Density Inhomogeneous Traffic Flow

### 3. 1. Introduction

In the preceding chapter it is shown that under Conditions (B), (C), (B') and (C'), the traffic flow becomes the inhomogeneous composed Poisson traffic flow. Not all the low density traffic flows, however, seem to satisfy the above conditions. Thus, these conditions are dropped in this chapter. To begin with, the low density traffic flow on an infinitely long road without intersections is dealt with and a sort of stochastic integral of the space process at the time origin is defined. Moreover it can be shown that all processes are expressed in terms of the stochastic integral. In Section 3. 3, the low density traffic flow on an finitely or infinitely long road with an intersection is dealt with and a stochastic integral of the time process at the space origin is defined. Moreover it can be shown that all processes are expressed in terms of the stochastic integrals of the space process at the time origin and the time process at the space origin. This implies that distributions of initial processes completely determine those of all processes. In Section 3. 4, velocity distributions are discussed. In Section 3. 5, results

obtained in the preceding sections are applied to the low density traffic flow with inhomogeneous Poisson initial processes. It may be noted that results of Weiss and Herman (1962) and Rényi (1964) are obtained as a special case.

### 3. 2. Traffic flow on an infinitely long road without intersections

The model to be discussed in this section is a one-way traffic flow satisfying the following condition  $(C_0)$  on an infinitely long road without intersections:

$(C_0)$ : individual vehicles travel at their own constant velocities (desired velocities) independently of any other vehicles.

This model is a frequently used one; it is thought that a traffic flow with relatively low density per lane satisfies Condition  $(C_0)$ . Throughout this chapter, vehicles are assumed to have no length.

Let  $R \times T \times V$  be a three-dimensional Euclidean subspace. Unless otherwise stated,  $R = (-\infty, \infty)$ ,  $T = (-\infty, \infty)$  and  $V = [0, \infty)$ . Denote by  $B(\cdot)$  an ordinary Borel field of subsets of  $\cdot$ . Let  $I \in B(R)$ ,  $J \in B(T)$  and  $K \in B(V)$ .

Throughout this chapter for  $r < s$ ,  $t < u$  and  $v < w$ ,  $I = [r, s)$ ,  $J = (t, u]$  and  $K = [v, w)$ , unless otherwise stated.

Let  $K^c$  be the complement of  $K$  on  $V$ . Moreover, the Lebesgue measure of  $\cdot$  is denoted by  $|\cdot|$ . Let  $(\underline{\Omega}, \underline{B}, \underline{P})$  be a probability space:  $\underline{\Omega}$  is an abstract space of points  $\omega$ ,  $\underline{B}$  is a  $\sigma$ -field of subsets of  $\underline{\Omega}$ , and  $\underline{P}$  is a probability measure. The family of the following random interval functions  $\{x_t(I, K), t \in T\}$  can be defined on  $(\underline{\Omega}, \underline{B}, \underline{P})$ :  
 $x_t(I, K)$ : the number of vehicles which exist at time  $t$

in a space interval  $I$  and have their own velocities belonging to a velocity interval  $K$ . The above defined process  $x_t(I, K)$  is called a space process at time  $t$ . It may be noted that in this chapter, random function and stochastic process are treated as synonymous. If we put  $x_t(r, K) = x_t(I, K)$  for  $0 < r$  and  $I = [0, r)$ , and  $x_t(r, K) = -x_t(I, K)$  for  $r < 0$  and  $I = [r, 0)$ , on the assumption of  $x_t(0, K) = 0$  almost surely (a.s.), then we have an ordinary point process  $x_t(r, K)$  for arbitrary fixed  $t \in T$  and  $K \in B(V)$ . Conversely,  $x_t(I, K)$  can be constructed by  $x_t(I, K) = x_t(s, K) - x_t(r, K)$ .

From the definition,  $x_t(I, K)$  is a non-negative integral valued  $\sigma$ -additive set function: that is, for  $I = \sum_i I_i$  and  $K = \sum_j K_j$ ,  $x_t(I, K) = \sum_{i,j} x_t(I_i, K_j)$  a.s., where  $I_i \cap I_j = K_i \cap K_j = \emptyset$  (empty set) for  $i \neq j$ . Therefore for  $I' \subset I$  and  $K' \subset K$ ,

$$(2.1) \quad x_t(I', K') \leq x_t(I', K) \leq x_t(I, K) \text{ a.s.}$$

The following abbreviations for  $x_t(I, K)$  are used: in the case of  $t=0$ ,  $x(I, K)$ ; in the case of  $K=V$ ,  $x_t(I)$ ; in the case of  $K=[0, v)$ ,  $x_t(I, v)$ .

Suppose that the distribution of the space process at the time origin  $\{x(I, K), I \in B(R), K \in B(V)\}$  is given or known. The distribution is called the initial distribution. The purpose of this section is to show that the initial distribution completely determines distributions of the

space process at time  $t(\neq 0)$  and all processes defined on  $(\Omega, \mathcal{B}, \mathcal{P})$  in the following:

$y_r(J, K)$ : the number of vehicles which pass through a point  $r$  during a time interval  $J$  and have their own velocities belonging to a velocity interval  $K$ .

$z(I, J, K)$ : the number of vehicles which exist or existed in a space interval  $I$  during a time interval  $J$  and have their own velocities belonging to a velocity interval  $K$ .

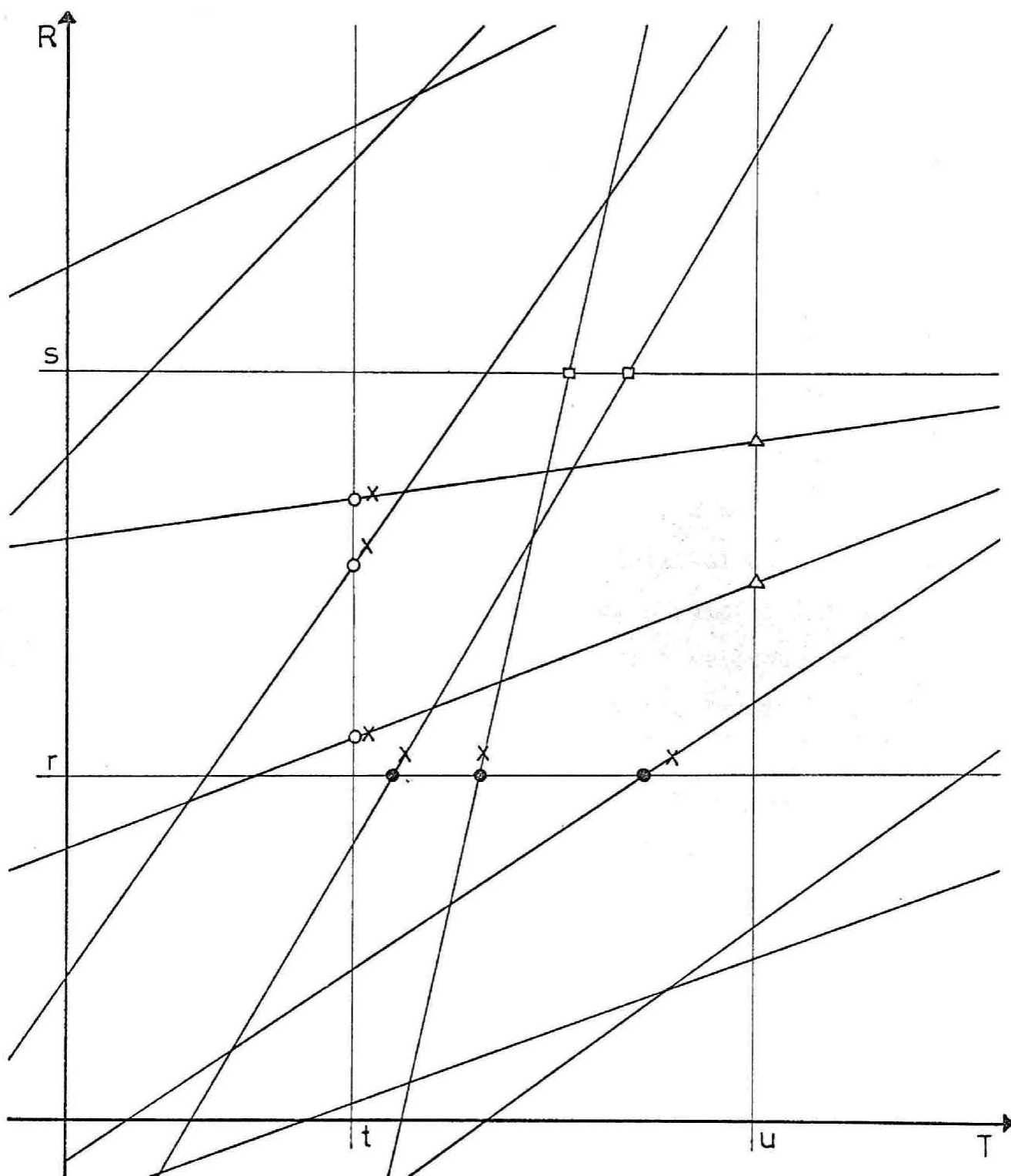
$x_{tu}(I, K)$ : the number of those vehicles with their own velocities belonging to a velocity interval  $K$  which exist in a space interval  $I$  not only at time  $t$  but also at time  $u(t \neq u)$ .

$y_{rs}(J, K)$ : the number of those vehicles with their own velocities belonging to a velocity interval  $K$  which pass through not only a point  $r$  but also a point  $s$  ( $r \neq s$ ) during a time interval  $J$ .

In comparison with  $x_t(I, K)$ ,  $y_r(J, K)$  is called a time process at point  $r$ , and the other processes subsidiary processes. Figure 1 shows the differences between the space process at time  $t$ , the time process at point  $r$ , and the subsidiary processes. Finally, observation processes are defined on the same probability space  $(\Omega, \mathcal{B}, \mathcal{P})$  as follows:

$X^-(J, K; v^*, r_0)$ : the number of those vehicles with their





$o: x_t(I, K), \bullet: y_r(J, K), x: z(I, J, K), \Delta: x_{tu}(I, K), \square: y_{rs}(J, K).$

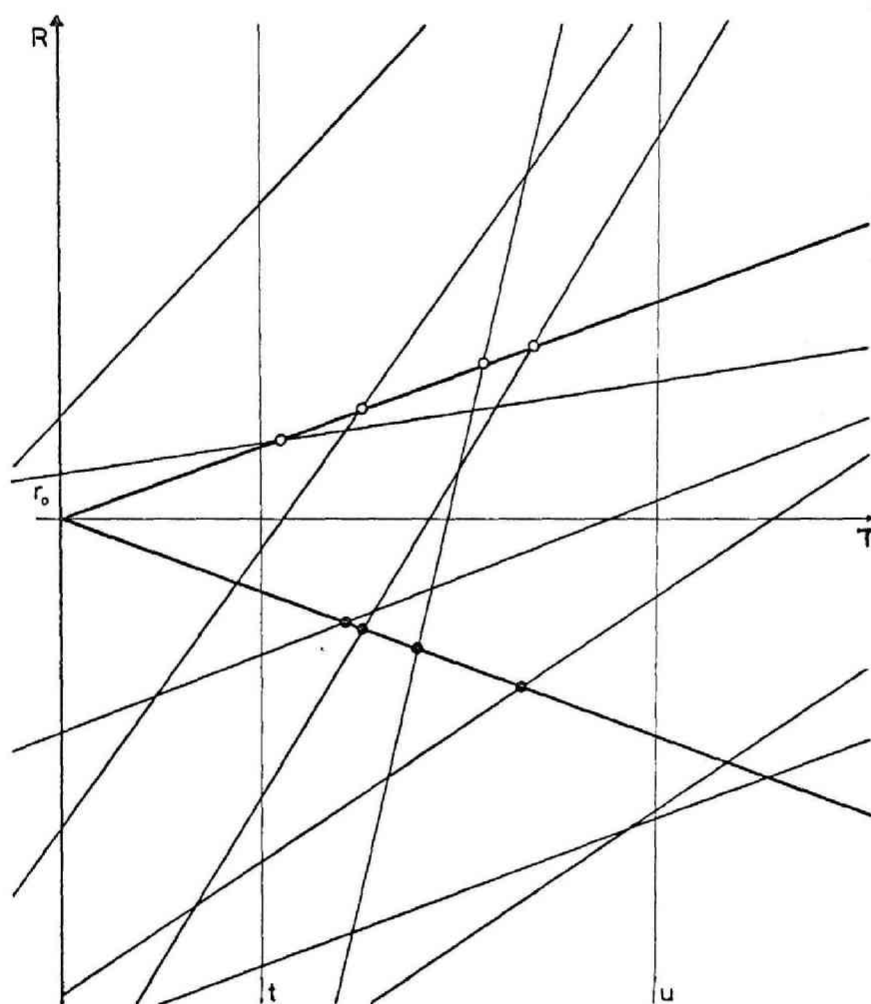
— : trajectories of vehicles.

FIG. 1

own velocities belonging to  $K$  which are observed during a time interval  $J$  by an observer who starts from a point  $r_0$  at time 0 and is moving with a velocity  $v^*$  in the opposite direction to the traffic flow.

$X^+(J, K; v^*, r_0)$ : the number of those vehicles with their own velocities belonging to  $K$  which are observed during a time interval  $J$  by an observer who starts from a point  $r_0$  at time 0 and is moving with a velocity  $v^*$  in the same direction as the traffic flow.

The differences between  $X^-(J, K; v^*, r_0)$  and  $X^+(J, K; v^*, r_0)$  are shown in Figure 2. An observation process represents observations on traffic flow by a moving observer with a constant velocity  $v^*$ . It should be noted that a vehicle is recorded when the observer catches up with it or is overtaken by it. On the other hand, a time process represents observations at a fixed point and a space process represents observations from the air. Subsidiary processes join observation processes to space processes, as shown below. It follows from the above definitions that all processes are non-negative integral valued  $\sigma$ -additive set functions and relations similar to (2.1)



$\circ : \hat{X}(J, K; \hat{V}^*, r_0), \quad \bullet : \bar{X}(J, K; \bar{V}^*, r_0).$   
 — : trajectories of vehicles.

FIG. 2

hold.

Traffic flow satisfying Condition  $(C_0)$  has the important property that all vehicles with a desired velocity  $v$  in a space interval  $I$  at time  $t(>0)$  are in the space interval  $I-tv=[r-tv, s-tv]$  at time zero. This property leads, on account of (2.1), to the following basic relation for space processes: for  $I=[r, s) \in B(R)$ ,  $K \in B(V)$ ,  $\hat{a}=\sup_{v \in K}(-tv)$ , and  $\tilde{a}=\inf_{v \in K}(-tv)$ ,

$$(2.2) \quad x([t+\hat{a}, s+\tilde{a}), K) \leq x_t(I, K) \leq x([r+\tilde{a}, s+\hat{a}), K) \text{ a.s.}$$

Similarly, the basic relations for the other processes are: for  $I=[r, s) \in B(R)$ ,  $J=(t, u] \in B(T)$ ,  $K \in B(V)$ ,  $\hat{a}=\sup_{v \in K}(-tv)$ ,  $\tilde{a}=\inf_{v \in K}(-tv)$ ,  $\hat{b}=\sup_{v \in K}(-uv)$  and  $\tilde{b}=\inf_{v \in K}(-uv)$ ,

$$(2.3) \quad x([r+\hat{b}, r+\tilde{a}), K) \leq y_r(J, K) \leq x([r+\tilde{b}, r+\hat{a}), K) \text{ a.s.,}$$

$$(2.4) \quad x([r+\hat{b}, s+\tilde{a}), K) \leq z(I, J, K) \leq x([r+\tilde{b}, s+\hat{a}), K) \text{ a.s.,}$$

$$(2.5) \quad x([r+\hat{a}, s+\tilde{b}), K) \leq x_{tu}(I, K) \leq x([r+\tilde{a}, s+\hat{b}), K) \text{ a.s.,}$$

and

$$(2.6) \quad x([s+\hat{b}, r+\tilde{a}), K) \leq y_{rs}(J, K) \leq x([s+\tilde{b}, r+\hat{a}), K) \text{ a.s.}$$

For example, Relation (2.3) results from the fact that all vehicles passing through a point  $r$  during a time interval  $J$  with a desired velocity  $v$  exist in the space interval  $[r-uv, r-tv)$  at time zero. The basic relations between observation processes and subsidiary processes are: for  $J=(t, u]$  and  $K \in B(V)$ ,

$$(2.7) \quad X^-(J, K; v^*, r_0) = z(I, J; K) \quad \text{a.s.},$$

where  $I = [r_0 - v^*u, r_0 - v^*t)$ ,

and

$$(2.8) \quad X^+(J, K; v^*, r_0) = x_{tu}(I, K) + y_{rs}(J, K) \quad \text{a.s.},$$

where  $r = r_0 + v^*t$ ,  $s = r_0 + v^*u$  and  $I = [r, s)$ .

In the above relation,  $x_{tu}(I, K)$  indicates the number of vehicles with which the observer catches up, and  $y_{rs}(J, K)$  indicates the number of vehicles which overtake the observer.

The property mentioned above ensures that the initial distribution completely determines distributions of all processes. The basic relations (2.2) to (2.6) suggest naturally that the following integral should be introduced. Denote by  $C(K)$  the set of real-valued uniformly continuous functions on  $K$ . It is to be noted that sample functions of  $x(I, v)$  are a.s. left-continuous non-decreasing step functions with respect to  $v$ . Therefore, for  $h(v)$ ,  $k(v) \in C(K)$ , the following integral can be defined as a usual Stieltjes integral if the last member of the following equation converges in one sense:

$$(D_1): \quad S([h(v), k(v)), K) = \int_K dx([h(v), k(v)), v) \\ \equiv \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} x([h(v'_{ni}), k(v'_{ni}))], K_{ni}),$$

where  $\{v_{ni}\}$  is an arbitrary partition of  $K$  such that

$$v=v_{n1} < \dots < v_{ni} < v_{ni+1} < \dots \quad v_{nk_{n+1}}=w$$

and

$$\max_i (v_{ni+1} - v_{ni}) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad K_{ni} = [v_{ni}, v_{ni+1})$$

$$\text{and } v'_{ni} \in K_{ni}.$$

Without loss of generality, it can be assumed that  $\{v_{n+1,i}\}$  is a subpartition of  $\{v_{ni}\}$ . It follows from the definition that if  $h(v) \geq k(v)$  for all  $v \in K$ , then  $S([h(v), k(v)), K) = 0$  a.s. Moreover, it is to be noted that  $h(v)$  and  $k(v)$  need not be uniformly continuous on  $K'$  such that  $x(I', K') = 0$  a.s. for  $K' \subset K$  and arbitrary  $I' \in \mathcal{B}(R)$ . Let  $\hat{h}_{ni}$  and  $\tilde{h}_{ni}$  be  $\sup_{v \in K_{ni}} h(v)$  and  $\inf_{v \in K_{ni}} h(v)$ , respectively. Denote by  $(C_1; K, h(v))$  the following condition:

$(C_1; K, h(v))$ : for  $K \in \mathcal{B}(V)$  and  $h(v) \in C(K)$ ,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} x([\tilde{h}_{ni}, \hat{h}_{ni}], K_{ni}) = 0 \text{ in probability.}$$

Conditions  $(C_1; K, h(v))$  and  $(C_1; K, k(v))$  are abbreviated to  $(C_1; K, h(v), k(v))$ .

Lemma 1.

Suppose that conditions  $(C_1; K, h(v), k(v))$  are satisfied.

Then, for  $K \in \mathcal{B}(V)$  and  $h(v), k(v) \in C(K)$ , as  $n \rightarrow \infty$ ,

$$\sum_{i=1}^{k_n} x([h(v'_{ni}), k(v'_{ni})], K_{ni}) \rightarrow S([h(v), k(v)], K) \text{ a.s.}$$

Proof.

The proof is straightforward. It follows from (2.1) that

for every  $n$  and  $i$ ,

$$\begin{aligned} x([\hat{h}_{ni}, \tilde{k}_{ni}), K_{ni}) &\leq x([h(v'_{ni}), k(v'_{ni})], K_{ni}) \\ &\leq x([\tilde{h}_{ni}, \hat{k}_{ni}), K_{ni}) \text{ a.s.} \end{aligned}$$

Therefore for every  $n$ ,

$$\begin{aligned} \tilde{S}_n = \sum_{i=1}^{k_n} x([\hat{h}_{ni}, \tilde{k}_{ni}), K_{ni}) &\leq S_n = \sum_{i=1}^{k_n} x([h(v'_{ni}), k(v'_{ni})], K_{ni}) \\ &\leq \hat{S}_n = \sum_{i=1}^{k_n} x([\tilde{h}_{ni}, \hat{k}_{ni}), K_{ni}) \text{ a.s.} \end{aligned}$$

Moreover, for arbitrary  $m$  and  $n$ ,  $\tilde{S}_m \leq \hat{S}_n$  a.s. As  $\hat{h}_{ni} \geq \tilde{k}_{ni}$  implies  $x([\hat{h}_{ni}, \tilde{k}_{ni}), K_{ni}) = 0$  a.s., without loss of generality it can be assumed that  $h(v) < k(v)$  for  $v \in K$ . Therefore for large  $n$  and arbitrary positive  $\varepsilon$ ,

$$\begin{aligned} P\left\{\sup_{m \geq n} |\hat{S}_m - \tilde{S}_m| \geq \varepsilon\right\} &= P\left\{\sup_{m \geq n} \sum_{i=1}^{k_m} (x([\tilde{h}_{mi}, \hat{h}_{mi}), K_{mi}) + x([\tilde{k}_{mi}, \hat{k}_{mi}), \right. \\ &\quad \left. K_{mi})) \geq \varepsilon\right\}. \end{aligned}$$

From Conditions  $(C_1; K, h(v), k(v))$ ,

$$\begin{aligned} P\left\{\sup_{m \geq n} |\hat{S}_m - \tilde{S}_m| \geq \varepsilon\right\} &\leq P\left\{\sup_{m \geq n} \sum_{i=1}^{k_m} x([\tilde{h}_{mi}, \hat{h}_{mi}), K_{mi}) \geq \varepsilon/2\right\} \\ &\quad + P\left\{\sup_{m \geq n} \sum_{i=1}^{k_m} x([\tilde{k}_{mi}, \hat{k}_{mi}), K_{mi}) \geq \varepsilon/2\right\} \\ &= P\left\{\sum_{i=1}^{k_n} x([\tilde{h}_{ni}, \hat{h}_{ni}), K_{ni}) \geq \varepsilon/2\right\} + P\left\{\sum_{i=1}^{k_n} x([\tilde{k}_{ni}, \hat{k}_{ni}), K_{ni}) \geq \varepsilon/2\right\} \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ .

Hence,  $S([h(v), k(v)), K)$  exists and

$$\mathbb{P}(\sup_{m \geq n} |S_m - S([h(v), k(v)), K])| \geq \varepsilon) \leq \mathbb{P}(\sup_{m \geq n} |\hat{S}_m - \tilde{S}_m| \geq \varepsilon) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

That is,  $S_n \rightarrow S([h(v), k(v)), K)$  a.s. The proof is concluded.

It may be noted that  $S([h(v), k(v)), K)$  is a  $\sigma$ -additive non-negative integral valued random set function with respect to  $K$  defined on  $(\Omega, \mathcal{B}, \mathbb{P})$ . Let  $p(z; x_t(I, K))$  and  $q(z; x_t(I, K))$  be a generating function of  $x_t(I, K)$  and its natural logarithm, respectively; that is,

$$p(z; \cdot) = \sum_{n=0}^{\infty} \mathbb{P}(\cdot = n) z^n$$

and

$$q(z; \cdot) = \log p(z; \cdot),$$

where  $|z| < 1$ .

#### Theorem 1.

Suppose that Condition  $(C_0)$  is satisfied. If Conditions  $(C_1; K, r-tv, s-tv)$  are satisfied, then for  $I=[r, s) \in \mathcal{B}(R)$ ,  $t \in T$  and  $K \in \mathcal{B}(V)$ ,

$$(2.9) \quad x_t(I, K) = S(I-tv, K) \text{ a.s.}$$

If Conditions  $(C_1; K, r-uv, r-tv)$  are satisfied, then for  $J=(t, u] \in \mathcal{B}(T)$ ,  $r \in R$  and  $K \in \mathcal{B}(V)$ ,

$$(2.10) \quad y_r(J, K) = S([r-uv, r-tv), K) \text{ a.s.}$$

Proof.

The same notation in the proof of Lemma 1 is used with  $h(v)$  and  $k(v)$  substituted by  $(r-tv)$  and  $(s-tv)$  respectively.



Clearly,

$$x_t(I, K) = \sum_{i=1}^{k_n} x_t(I, K_{ni}) \text{ a.s.}$$

Therefore, on account of (2.2), for every  $n$ ,  $\tilde{S}_n \leq x_t(I, K) \leq \hat{S}_n$  a.s., while, for every  $n$ ,  $\tilde{S}_n \leq S_n \leq \hat{S}_n$  a.s. Hence, similarly to the proof of Lemma 1,

$$\mathbb{P}\left\{\sup_{m \geq n} |S_m - x_t(I, K)| \geq \varepsilon\right\} \leq \mathbb{P}\left\{\sup_{m \geq n} |\hat{S}_m - \tilde{S}_m| \geq \varepsilon\right\} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

That is,  $S_n \rightarrow x_t(I, K)$  a.s. It follows from Lemma 1 that  $S_n \rightarrow S(I - tv, K)$  a.s. The proof of (2.9) is concluded. Similarly, (2.10) can be proved on account of (2.3). The proof is omitted.

Theorem 1 shows immediately that

$$p(z; x_t(I, K)) = p(z; S(I - tv, K))$$

and

$$p(z; y_r(J, K)) = p(z; S([r - uv, r - tv], K)).$$

The analogous results for subsidiary processes follow from Relations (2.4) to (2.6). Since the proof is entirely similar to that of Theorem 1, it is omitted.

#### Theorem 2.

Suppose that Condition  $(C_0)$  is satisfied. If Conditions  $(C_1; K, r - uv, s - tv)$  are satisfied, then for  $I = [r, s] \in B(R)$ ,  $J = (t, u] \in B(T)$  and  $K \in B(V)$ ,

$$(2.11) \quad z(I, J, K) = S([r-uv, s-tv], K) \text{ a.s.}$$

If Conditions  $(C_1; K \wedge K_1, r-tv, s-uv)$  are satisfied, then for  $I=[r, s] \in B(R), J=(t, u] \in B(T)$  and  $K \in B(V)$ ,

$$(2.12) \quad x_{tu}(I, K) = S([r-tv, s-uv], K \wedge K_1) \text{ a.s.},$$

where

$$K_1 = [0, |I|/|J|].$$

If Conditions  $(C_1; K \wedge K_1^c, s-uv, r-tv)$  are satisfied, then for  $I=[r, s] \in B(R), J=(t, u] \in B(T)$  and  $K \in B(V)$ ,

$$(2.13) \quad y_{rs}(J, K) = S([s-uv, r-tv], K \wedge K_1^c) \text{ a.s.},$$

where

$$K_1 = [0, |I|/|J|].$$

Now Figure 2 suggests that

$$(2.14) \quad X^-(J, K; v^*, r_0) = X^+(J, K; -v^*, r_0) \text{ a.s.}$$

In fact, it follows from exchanging  $r$  and  $s$  in (2.12) and (2.13) that formally, for  $r < s$ ,

$$(2.15) \quad x_{tu}([s, r], K) = 0 \text{ a.s.}$$

and

$$(2.16) \quad y_{sr}(J, K) = z([r, s], J, K) \text{ a.s.},$$

because  $K_1 = \emptyset$ . That is, for  $s < r$ ,

$$x_{tu}([r, s], K) = 0 \text{ a.s.}$$

and

$$y_{rs}(J,K)=z([s,r),J,K) \text{ a.s.}$$

Therefore, substituting  $(-v^*)$  for  $v^*$  in (2.8), for  $r=r_0-v^*t > s=r_0-v^*u$ ,

$$\begin{aligned} X^+(J,K;-v^*,r_0) &= y_{rs}(J,K) = z([s,r),J,K) \\ &= X^-(J,K;v^*,r_0) \text{ a.s.} \end{aligned}$$

Hence, in the following, superscripts  $+$  and  $-$  are dropped, unless it is necessary to distinguish them. The following theorem follows directly from Theorem 2 and (2.8).

Theorem 3.

Suppose that conditions of Theorem 2 are satisfied for  $r=r_0+v^*t$  and  $s=r_0+v^*u$ . Then, for  $J=(t,u] \in B(T)$ ,  $K \in B(V)$ ,  $v^* \in (-\infty, \infty)$  and  $K^*=[0, v^*]$ ,

$$\begin{aligned} X(J,K;v^*,r_0) &= S([r_0+(v^*-v)t, r_0+(v^*-v)u), K \cap K^*) \\ &\quad + S([r_0+(v^*-v)u, r_0+(v^*-v)t), K \cap (K^*)^c) \text{ a.s.} \end{aligned}$$

This theorem implies that distributions of observation processes are completely determined by the initial distribution.

### 3. 3. Generalization to traffic flow with influx

A traffic flow on an infinitely long road  $R = (-\infty, \infty)$  without intersections has been discussed in the preceding section. The model to be discussed in this section is a one-way traffic flow satisfying Condition  $(C_0)$  on a road  $R = [0, R')$  with an intersection at the space origin for a positive number  $R'$  finite or not. That is, the traffic flow to be discussed has influx at the space origin. In this case, it is impossible to express all processes in terms of the space process at the time origin as in the preceding section. Therefore, in this section, suppose that distributions of the space process at the time origin  $\{x(I, K), I \in B(R), K \in B(V)\}$  and the time process at the space origin  $\{y(J, K), J \in B(T), K \in B(V)\}$  are known or given. The distributions are called the initial distributions. The purpose of this section is to show that the initial distributions completely determine distributions of all processes defined in the preceding section. It can be shown, as a special case, in Theorem 7 that the distribution of the time process at the space origin completely determines those of all processes. In order to reduce formal complexities, let  $T = [0, T')$  for a positive number  $T'$  finite or not: the case of  $T = (-\infty, T')$  is discussed in Theorem 7. That is, in this section,

$R = [0, R')$ ,  $T = [0, T')$  and  $V = [0, \infty)$ , unless otherwise stated. Throughout this section, let  $K_{1x} = [0, r/t]$ ,  $K_{2x} = (s/t, \infty)$ ,  $K_{1y} = [0, r/u]$ ,  $K_{2y} = (r/t, \infty)$ ,  $K_1 = [0, (s-r)/(u-t)]$  and  $K_2 = [0, s/u]$ . The basic relations for space processes, time processes and subsidiary processes are: for  $I = [r, s) \in B(R)$ ,  $J = (t, u] \in B(T)$  and  $K = [v, w) \in B(V)$ .

$$(3.1) \quad \begin{aligned} x([r-tv, s-tw), K \cap K_{1x}) &\leq x_t(I, K \cap K_{1x}) \\ &\leq x([r-tw, s-tv), K \cap K_{1x}) \text{ a.s.,} \end{aligned}$$

$$(3.2) \quad \begin{aligned} y((t-s/w, t-r/v], K \cap K_{2x}) &\leq x_t(I, K \cap K_{2x}) \\ &\leq y((t-s/v, t-r/w], K \cap K_{2x}) \text{ a.s.,} \end{aligned}$$

$$(3.3) \quad \begin{aligned} x([r-uv, r-tw), K \cap K_{1y}) &\leq y_r(J, K \cap K_{1y}) \\ &\leq x([r-uw, r-tv), K \cap K_{1y}) \text{ a.s.,} \end{aligned}$$

$$(3.4) \quad \begin{aligned} y((t-r/w, u-r/v], K \cap K_{2y}) &\leq y_r(J, K \cap K_{2y}) \\ &\leq y((t-r/v, u-r/w], K \cap K_{2y}) \text{ a.s.} \end{aligned}$$

$$(3.5) \quad \begin{aligned} x([r-uv, s-tw), K \cap K_{1y}) &\leq z(I, J, K \cap K_{1y}) \\ &\leq x([r-uw, s-tv), K \cap K_{1y}) \text{ a.s.} \end{aligned}$$

$$(3.6) \quad \begin{aligned} y((t-s/w, u-r/v], K \cap K_{2x}) &\leq z(I, J, K \cap K_{2x}) \\ &\leq y((t-s/v, u-r/w], K \cap K_{2x}) \text{ a.s.,} \end{aligned}$$

$$(3.7) \quad \begin{aligned} x([r-tv, s-uw), K \cap K_{1x} \cap K_1) &\leq x_{tu}(I, K \cap K_{1x} \cap K_1) \\ &\leq x([r-tw, s-uv), K \cap K_{1x} \cap K_1) \text{ a.s.,} \end{aligned}$$

$$(3.8) \quad \begin{aligned} y((u-s/w, t-r/v], K \cap K_1 \cap K_2^c) &\leq x_{tu}(I, K \cap K_{1x} \cap K_2^c) \\ &\leq y((u-s/v, t-r/w], K \cap K_1 \cap K_2^c) \text{ a.s.} \end{aligned}$$

$$(3.9) \quad \begin{aligned} x([s-uv, r-tw), K \cap K_1^c \cap K_2) &\leq y_{rs}(J, K \cap K_1^c \cap K_2) \\ &\leq x([s-uw, r-tv), K \cap K_1^c \cap K_2) \text{ a.s.,} \end{aligned}$$

and

$$(3.10) \quad y((t-r/w, u-s/v], K \cap K_{2y} \cap K_1^C) \leq y_{rs}(J, K \cap K_{2y} \cap K_1^C)$$

$$\leq y((t-r/v, u-s/w], K \cap K_{2y} \cap K_1^C) \text{ a.s.}$$

if  $K \cap K_{..} \neq \emptyset$  and  $K \not\subset K_{..}$ , then in the above relations,  $v$  or  $w$  is replaced by an end point of the interval  $K_{..}$ , where  $K_{..}$  represents  $K_{1x}$  and  $K_1 \cap K_2^C$ , etc.

Similarly to  $(D_1)$  the following integral is defined for  $K \in B(V)$  and  $h(v), k(v) \in C(K)$ :

$$(D_1'): \quad U((h(v), k(v)], K) = \int_K dy((h(v), k(v)], v) \\ \equiv \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} y((h(v'_{ni}), k(v'_{ni})), K_{ni}).$$

Denote by  $(C_1'; K, h(v))$  the following condition:

$(C_1'; K, h(v))$ : for  $K \in B(V)$  and  $h(v) \in C(K)$ ,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} y((\tilde{h}_{ni}, \tilde{h}_{ni}], K_{ni}) = 0 \text{ in probability.}$$

Conditions  $(C_1'; K, h(v))$  and  $(C_1'; K, k(v))$  are abbreviated to  $(C_1'; K, h(v), k(v))$ . The following lemma can be derived in a similar way to Lemma 1. The proof is omitted to avoid duplications.

Lemma 2. Suppose that Conditions  $(C_1'; K, h(v), k(v))$  are satisfied. Then, for  $K \in B(V)$  and  $h(v), k(v) \in C(K)$ , as  $n \rightarrow \infty$ ,

$$\sum_{i=1}^{k_n} y((h(v'_{ni}), k(v'_{ni})), K_{ni}) \rightarrow U((h(v), k(v)], K) \text{ a.s.}$$

On the basis of Relations (3.1) to (3.4), it follows from Lemmas 1 and 2 that the following lemma can be proved in a similar manner to that of Theorem 1. The

proof is omitted to avoid duplications.

Lemma 3. Suppose that Condition  $(C_0)$  is satisfied. If Conditions  $(C_1; K \cap K_{1x}, r-tv, s-tv)$  are satisfied, then for  $I = [r, s) \in B(R)$ ,  $t \in T$  and  $K \in B(V)$ ,

$$x_t(I, K \cap K_{1x}) = S(I-tv, K \cap K_{1x}) \text{ a.s.}$$

If Conditions  $(C'_1; K \cap K_{2x}, t-s/v, t-r/v)$  are satisfied, then for  $I = [r, s) \in B(R)$ ,  $t \in T$  and  $K \in B(V)$ ,

$$x_t(I, K \cap K_{2x}) = U((t-s/v, t-r/v], K \cap K_{2x}) \text{ a.s.}$$

If Conditions  $(C_1; K \cap K_{1y}, r-uv, r-tv)$  are satisfied, then for  $J = (t, u] \in B(T)$ ,  $r \in R$  and  $K \in B(V)$ ,

$$y_r(J, K \cap K_{1y}) = S([r-uv, r-tv), K \cap K_{1y}) \text{ a.s.}$$

If Conditions  $(C'_1; K \cap K_{2y}, t-r/v, u-r/v)$  are satisfied, then for  $J = (t, u] \in B(T)$ ,  $r \in R$  and  $K \in B(V)$ ,

$$y_r(J, K \cap K_{2y}) = U(J-r/v, K \cap K_{2y}) \text{ a.s.}$$

Moreover, the following lemma is prepared for Theorem 4.  
Let

$$K_{3x} = (K_{1x} \cup K_{2x})^c = (r/t, s/t] \text{ and } K_{3y} = (K_{1y} \cup K_{2y})^c = (r/u, r/t].$$

Lemma 4. Suppose that Condition  $(C_0)$  is satisfied. If Conditions  $(C_1; K \cap K_{3x}, s-tv)$  and  $(C'_1; K \cap K_{3x}, t-r/v)$  are satisfied, then for  $I = [r, s) \in B(R)$ ,  $t \in T$  and  $K \in B(V)$ .

$$x_t(I, K \cap K_{3x}) = S([0, s-tv), K \cap K_{3x}) + U((0, t-r/v], K \cap K_{3x}) \text{ a.s.}$$

If Conditions  $(C_1; K \cap K_{3y}, r - tv)$  and  $(C'_1; K \cap K_{3y}, u - r/v)$  are satisfied, then for  $J = (t, u] \in B(T)$ ,  $r \in R$  and  $K \in B(V)$ ,

$$y_r(J, K \cap K_{3y}) = S([0, r - tv), K \cap K_{3y}) + U((0, u - r/v], K \cap K_{3y}) \text{ a.s.}$$

Proof.

It is obvious that for  $v \in K_{3x}$ ,  $r - vt \leq s$  and for  $v \in K_{3y}$ ,  $tv \leq r < uv$ . Similarly to Relations (3.1), (3.2), (3.3) and (3.4), the following basic relations hold: for every  $n$  and  $i$ ,

$$\begin{aligned} & x([0, s - tv_{ni+1}), K_{ni} \cap K_{3x}) + y((0, t - r/v_{ni}], K_{ni} \cap K_{3x}) \\ & \leq x_t(I, K_{ni} \cap K_{3x}) \leq x([0, s - tv_{ni}), K_{ni} \cap K_{3x}) + y((0, t - r/v_{ni+1}], \\ & \quad K_{ni} \cap K_{3x}) \text{ a.s.} \end{aligned}$$

and

$$\begin{aligned} & x([0, r - tv_{ni+1}), K_{ni} \cap K_{3y}) + y((0, u - r/v_{ni}], K_{ni} \cap K_{3y}) \\ & \leq y_r(J, K_{ni} \cap K_{3y}) \leq x([0, r - tv_{ni}), K_{ni} \cap K_{3y}) + y((0, u - r/v_{ni+1}], \\ & \quad K_{ni} \cap K_{3y}) \text{ a.s.,} \end{aligned}$$

where for  $K_{ni}$  such that  $K_{ni} \cap K_3 \neq \emptyset$  and  $K_{ni} \not\subset K_3$ ,  $v_{ni}$  or  $v_{ni+1}$  is replaced by an end point of  $K_3$ .

Let

$$\begin{aligned} S_n(r) &= \sum_{i=1}^{k_n} x([0, r - tv'_{ni}), K_{ni} \cap K_{3x}), \\ U_n(t) &= \sum_{i=1}^{k_n} y((0, t - r/v'_{ni}], K_{ni} \cap K_{3y}), \\ \tilde{S}_n(r) &= \sum_{i=1}^{k_n} x([0, r - tv_{ni+1}), K_{ni} \cap K_{3x}), \\ \tilde{U}_n(t) &= \sum_{i=1}^{k_n} y((0, t - r/v_{ni}], K_{ni} \cap K_{3y}), \\ \hat{S}_n(r) &= \sum_{i=1}^{k_n} x([0, r - tv_{ni}), K_{ni} \cap K_{3x}) \end{aligned}$$



and

$$\hat{U}_n(t) = \sum_{i=1}^{k_n} y((0, t-r/v_{ni+1}], K_{ni} \cap K_{3y}).$$

Then, for every  $n$ ,

$$\tilde{S}_n(s) + \tilde{U}_n(t) \leq x_t(I, K \cap K_{3x}) \leq \hat{S}_n(s) + \hat{U}_n(t) \text{ a.s.}$$

and

$$\tilde{S}_n(r) + \tilde{U}_n(u) \leq y_r(J, K \cap K_{3y}) \leq \hat{S}_n(r) + \hat{U}_n(u) \text{ a.s.}$$

It follows from Conditions  $(C_1; K \cap K_{3x}, s-tv)$  and  $(C_1; K \cap K_{3x}, t-r/v)$  that

$$\begin{aligned} & P \left\{ \sup_{m \geq n} |S_m(s) + U_m(t) - x_t(I, K \cap K_{3x})| \geq \varepsilon \right\} \\ & \leq P \{ |\hat{S}_n(s) - \tilde{S}_n(s) + \hat{U}_n(t) - \tilde{U}_n(t)| \geq \varepsilon \} \\ & \leq P \{ |\hat{S}_n(s) - \tilde{S}_n(s)| \geq \varepsilon/2 \} + P \{ |\hat{U}_n(t) - \tilde{U}_n(t)| \geq \varepsilon/2 \} \rightarrow 0, \\ & \text{as } n \rightarrow \infty. \end{aligned}$$

That is,  $S_n(s) + U_n(t) \rightarrow x_t(I, K \cap K_{3x})$  a.s. Similarly,

$$S_n(r) + U_n(u) \rightarrow y_r(J, K \cap K_{3y}) \text{ a.s.}$$

While Lemmas 1 and 2 show that

$$\begin{aligned} S_n(s) + U_n(t) & \rightarrow S([0, s-tv), K \cap K_{3x}) + U((0, t-r/v], \\ & K \cap K_{3x}) \text{ a.s.} \end{aligned}$$

and

$$\begin{aligned} S_n(r) + U_n(u) & \rightarrow S([0, r-tv), K \cap K_{3y}) + U((0, u-r/v], \\ & K \cap K_{3y}) \text{ a.s.} \end{aligned}$$

Hence the proof is concluded.

By use of Lemmas 3 and 4, it can be shown that the space process at time  $t(\in T)$  and the time process at point  $r(\in R)$  are expressed in terms of the space process at the time origin and the time process at the space origin. Therefore, distributions of  $x_t(I, K)$  and  $y_r(J, K)$  are completely determined by the initial distributions.

Theorem 4. Suppose that Condition  $(C_0)$  is satisfied. If Conditions  $(C_1; K \cap K_{1x}, r-tv, s-tv)$ ,  $(C'_1; K \cap K_{2x}, t-s/v, t-r/v)$ ,  $(C_1; K \cap K_{3x}, s-tv)$  and  $(C'_1; K \cap K_{3x}, t-r/v)$  are satisfied, then for  $I = [r, s] \in B(R)$ ,  $t \in T$  and  $K \in B(V)$

$$\begin{aligned} x_t(I, K) = & S(I-tv, K \cap K_{1x}) + S([0, s-tv], K \cap K_{3x}) \\ (3.11) \quad & + U((0, t-r/v], K \cap K_{3x}) + U((t-s/v, t-r/v], \\ & K \cap K_{2x}) \text{ a.s.,} \end{aligned}$$

where  $K_{1x} = [0, r/t]$ ,  $K_{2x} = (s/t, \infty)$  and  $K_{3x} = (r/t, s/t]$ .

If Conditions  $(C_1; K \cap K_{1y}, r-uv, r-tv)$ ,  $(C'_1; K \cap K_{2y}, t-r/v, u-r/v)$ ,  $(C_1; K \cap K_{3y}, r-tv)$  and  $(C'_1; K \cap K_{3y}, u-r/v)$  are satisfied, then for  $J = (t, u] \in B(T)$ ,  $r \in R$  and  $K \in B(V)$ ,

$$\begin{aligned} y_r(J, K) = & S([r-uv, r-tv], K \cap K_{1y}) + S([0, r-tv], K \cap K_{3y}) \\ (3.12) \quad & + U((0, u-r/v], K \cap K_{3y}) + U(J-r/v, K \cap K_{2y}) \text{ a.s.} \end{aligned}$$

where  $K_{1y} = [0, r/u]$ ,  $K_{2y} = (r/t, \infty)$  and  $K_{3y} = (r/u, r/t]$ .

Proof.

Clearly,

$$x_t(I, K) = \sum_{j=1}^3 x_t(I, K \cap K_{jx}) \text{ a.s.}$$

and

$$y_r(J, K) = \sum_{j=1}^3 y_r(J, K \cap K_{jy}) \text{ a.s.}$$

Hence, this theorem is directly deduced from Lemmas 3 and 4, and the proof is concluded.

Similarly, the following theorem can be derived from Relations (3.5) to (3.10). The proof is omitted to avoid duplications.

Theorem 5. Suppose that Condition  $(C_0)$  is satisfied.

If Conditions  $(C_1; K \cap K_{1y}, r-uv, s-tv)$ ,  $(C'_1; K \cap K_{2x}, t-s/v, u-r/v)$ ,  $(C_1; K \cap K_{1y}^c \cap K_{2x}^c, s-tv)$  and  $(C'_1; K \cap K_{1y}^c \cap K_{2x}^c, u-r/v)$  are satisfied, then for  $I = [r, s) \in B(R)$ ,  $J = (t, u] \in B(T)$  and  $K \in B(V)$ ,

$$\begin{aligned} z(I, J, K) &= S([r-uv, s-tv], K \cap K_{1y}) + S([0, s-tv], K \cap K_{1y}^c \cap K_{2x}^c) \\ (3.13) \quad &+ U((0, u-r/v], K \cap K_{1y}^c \cap K_{2x}^c) + U((t-s/v, u-r/v], \\ &K \cap K_{2x}) \text{ a.s.,} \end{aligned}$$

where  $K_{1y} = [0, r/u]$  and  $K_{2x} = (s/t, \infty)$ .

If Conditions  $(C_1; K \cap K_{1x} \cap K_1, r-tv, s-uv)$ ,  $(C'_1; K \cap K_1 \cap K_2^c, u-s/v, t-r/v)$ ,  $(C_1; K \cap K_{1x}^c \cap K_1 \cap K_2, s-uv)$  and  $(C'_1; K \cap K_{1x}^c \cap K_1 \cap K_2, t-r/v)$  are satisfied, then for  $I = [r, s) \in B(R)$ ,  $J = (t, u] \in B(T)$  and  $K \in B(V)$ ,

(3.14)

$$x_{tu}(I, K) = S([r-tv, s-uv], K \cap K_{1x} \cap K_1) + S([0, s-uv], K \cap K_{1x}^c \cap K_1 \cap K_2)$$

$$+ U((0, t-r/v], K \cap K_{1x}^c \cap K_1 \cap K_2) + U((u-s/v, t-r/v], K \cap K_1 \cap K_2^c) \text{ a.s.},$$

where  $K_{1x} = [0, r/t]$ ,  $K_1 = [0, |I|/|J|]$  and  $K_2 = [0, s/u]$ .

If Conditions  $(C_1; K \cap K_1^c \cap K_2, s-uv, r-tv)$ ,  $(C_1'; K \cap K_{1x}^c \cap K_1^c, t-r/v, u-s/v)$ ,  $(C_1; K \cap K_{1x} \cap K_1^c \cap K_2^c, r-tv)$  and  $(C_1'; K \cap K_{1x} \cap K_1^c \cap K_2^c, u-s/v)$  are satisfied, then for  $I = [r, s) \in B(R)$ ,  $J = (t, u] \in B(T)$  and  $K \in B(V)$ ,

(3.15)

$$y_{rs}(J, K) = S([s-uv, r-tv], K \cap K_1^c \cap K_2) + S([0, r-tv], K \cap K_{1x} \cap K_1^c \cap K_2^c) + U((0, u-s/v], K \cap K_{1x} \cap K_1^c \cap K_2^c) + U((t-r/v, u-s/v], K \cap K_{1x}^c \cap K_1^c) \text{ a.s.},$$

where  $K_{1x} = [0, r/t]$ ,  $K_1 = [0, |I|/|J|]$  and  $K_2 = [0, s/u]$ .

Relations (2.15) and (2.16) are derived from interchanging  $r$  and  $s$  in (3.14) and (3.15). Therefore, (2.14) holds. Combination of Relations (2.8), (3.14) and (3.15) leads directly to the following theorem.

Theorem 6. Suppose that conditions of Theorem 5 are satisfied for  $r = r_0 + v^*t$  and  $s = r_0 + v^*u$ . Then, for  $J = (t, u] \in B(T)$ ,  $K \in B(V)$ ,  $v^* \in (-\infty, \infty)$  and  $r_0$  such that  $(r_0 + v^*u) \in R$ ,

$$\begin{aligned} X(J, K; v^*, r_0) &= S([r_0 + (v^*-v)t, r_0 + (v^*-v)u], K \cap K_{1x} \cap K^*) \\ &\quad + S([0, r_0 + (v^*-v)u], K \cap K_{1x}^c \cap K_{2x}^c \cap K^*) \\ &\quad + U((0, t-(r_0 + v^*t)/v], K \cap K_{1x}^c \cap K_{2x}^c \cap K^*) \\ &\quad + U((u-(r_0 + v^*u)/v, t-(r_0 + v^*t)/v], K \cap K_{2x} \cap K^*) \end{aligned} \quad (3.16)$$

$$\begin{aligned}
& +S([r_0 - (v-v^*)u, r_0 - (y-v^*)t], K \cap K_{2X}^c \cap (K^*)^c) \\
& +S([0, r_0 - (v-v^*)t], K \cap K_{1X} \cap K_{2X} \cap (K^*)^c) \\
& +U((0, u - (r_0 + v^*u)/v], K \cap K_{1X} \cap K_{2X} \cap (K^*)^c) \\
& +U((t - (r_0 + v^*t)/v, u - (r_0 + v^*u)/v], K \cap K_{1X}^c \cap (K^*)^c) \text{ a.s.},
\end{aligned}$$

where  $K_{1X} = [0, v^* + r_0/t]$ ,  $K_{2X} = (v^* + r_0/u, \infty)$  and  $K^* = [0, v^*]$ .

It may be noted that if  $r_0 = 0$ , then Relation (3.16) is reduced to the following relation:

$$\begin{aligned}
X(J, K; v^*, 0) &= S([(v^* - v)t, (v^* - v)u], K \cap K^*) \\
&+ U((t(v-v^*)/v, u(v-v^*)/v], K \cap (K^*)^c) \text{ a.s.}
\end{aligned}$$

Finally, if  $y(J, K)$  is defined for  $T = (-\infty, T')$ ,  $J \in B(T)$  and  $K \in B(V)$ , then the above Theorems 4 and 6 are easily reduced to the following theorem.

Theorem 7. Let  $T = (-\infty, T')$ ,  $I = [r, s] \in B(R)$ ,  $J = (t, u) \in B(T)$  and  $K \in B(V)$ . Suppose that Condition  $(C_0)$  is satisfied and  $y(J, K') = 0$  a.s., for arbitrary  $J \in B(T)$  and  $K' = [0, \epsilon]$ , where  $\epsilon$  is a small positive number. If Conditions  $(C_1^I; K, t - s/v, t - r/v)$  are satisfied, then

$$(3.17) \quad x_t(I, K) = U((t-s/v, t-r/v], K) \text{ a.s.}$$

If Conditions  $(C_1^I; K, t - r/v, u - r/v)$  are satisfied, then

$$(3.18) \quad y_r(J, K) = U(J - r/v, K) \text{ a.s.}$$

If Conditions  $(C_1^I; K \cap K^*, u - (r_0 + v^*u)/v, t - (r_0 + v^*t)/v)$  and  $(C_1^I; K \cap (K^*)^c, t - (r_0 + v^*t)/v, u - (r_0 + v^*u)/v)$  are

satisfied, then

$$\begin{aligned}
 (3.19) \quad X(J, K; v^*, r_0) &= U((u - (r_0 + v^*u)/v, t - (r_0 + v^*t)/v], K \cap K^*) \\
 &+ U((t - (r_0 + v^*t)/v, u - (r_0 + v^*u)/v], \\
 &\quad K \cap (K^*)^c) \text{ a.s.},
 \end{aligned}$$

where  $K^* = [0, v^*]$ .

### 3. 4. Velocity distributions

$\{x_t(I, K), I \in B(R), K \in B(V), t \in T\}$  can describe all possible states of traffic flow, but information on velocities is implicit. This fact may be improved to some extent by introducing a space velocity distribution function (s.v.d.f.). Let  $F_t(v, I)$  be an s.v.d.f. An s.v.d.f.  $F_t(v, I)$  is regarded as the probability that any vehicle in a space interval  $I$  at time  $t$  has slower velocity than  $v$ . If  $F_t(v, I)$  has the above mentioned property, then

$$\begin{aligned} p(z; x_t(I, K)) &= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \binom{m}{n} \left( \int_K dF_t(v, I) \right)^n \left( 1 - \int_K dF_t(v, I) \right)^{m-n} \\ &\quad \cdot \underline{P}(x_t(I) = m) z^n \\ &= \sum_{m=0}^{\infty} \left\{ 1 + (z-1) \int_K dF_t(v, I) \right\}^m \cdot \underline{P}(x_t(I) = m) \\ &= p(1 + (z-1) \int_K dF_t(v, I); x_t(I)); \end{aligned}$$

that is,

$$(E): \quad p(z; x_t(I, K)) = p\left(1 + (z-1) \int_K dF_t(v, I); x_t(I)\right)$$

and vice versa. By differentiating the above relation, the following relation is obtained, if  $\underline{E}(x_t(I)) < \infty$ :

$$(D_2): \quad \underline{E}(x_t(I, v)) = F_t(v, I) \underline{E}(x_t(I)),$$

where  $\underline{E}(\cdot)$  denotes an expectation of  $\cdot$ .

Clearly, an s.v.d.f. defined by (E), which is the

conventional definition, necessarily has property  $(D_2)$ . In this chapter, however, an s.v.d.f.  $F_t(v, I)$  is defined as  $(D_2)$ : since  $\underline{E}(x_t(I, v))$  is a non-negative valued increasing function in  $v$ ,  $F_t(v, I)$  defined by  $(D_2)$  satisfies properties of a d.f., except the case of  $\underline{E}(x_t(I)) = 0$ , where  $F_t(v, I)$  is defined as a suitable function satisfying properties of an ordinary d.f. Because, in general, an s.v.d.f.  $F_t(v, I)$  for traffic flow satisfying Condition  $(C_0)$  cannot have property  $(E)$  for all  $t \in T$ . For illustration, suppose that the initial distribution is composed of the distribution of  $x(I)$  and the s.v.d.f. with property  $(E)$ . Then the initial distribution completely determines the distributions of  $x_t(I)$  and  $x_t(I, v)$ , as shown in (2.9). Therefore, in general,  $F_t(v, I)$  determined by  $(D_2)$  cannot satisfy  $(E)$ . A simple example is given in the Appendix I. It may be noted that as an exceptional case, Poisson traffic flow has an s.v.d.f. with property  $(E)$  for all  $t \in T$ . As the natural continuation of definition  $(D_2)$ , an infinitesimal s.v.d.f.  $F_t(v, r)$  is defined as follows

$$(D_3): \quad F_t(v, r) = \lim_{s \rightarrow r} F_t(v, I), \text{ for } I = [r, s).$$

If  $\underline{E}(x_t(I))$  is absolutely continuous with respect to Lebesgue measure on  $R$ , then  $F_t(v, r)$  exists almost everywhere (a.e.) on  $R$ .



Similarly, a time velocity distribution function (t.v.d.f.)  $G_r(v, J)$  is defined as follows:

$$(D'_2): \quad G_r(v, J) \mathbb{E}(y_r(J)) = \mathbb{E}(y_r(J, v));$$

if  $\mathbb{E}(y_r(J)) = 0$ , then  $G_r(v, J)$  is defined as a suitable function with properties of an ordinary distribution function. If a t.v.d.f.  $G_r(v, J)$  satisfies the relation

$$p(z; y_r(J, K)) = p\left(1 + (z-1) \int_K dG_r(v, J); y_r(J)\right),$$

then it is said that  $G_r(v, J)$  has property (E'). Moreover, an infinitesimal t.v.d.f.  $g_r(v, t)$  is defined as follows:

$$(D'_3) \quad g_r(v, t) = \lim_{t' \rightarrow t} G_r(v, J), \text{ for } J = (t', t].$$

If  $\mathbb{E}(y_r(J))$  is absolutely continuous with respect to Lebesgue measure on  $T$ , then  $g_r(v, t)$  exists a.e. on  $T$ .

### 3. 5. Application to inhomogeneous Poisson traffic flow

In this section, traffic flow with influx is dealt with in view of generality. To begin with, sufficient conditions are shown under which the space process at the time origin is restricted to an inhomogeneous Poisson process. Rényi (1951) has shown sufficient conditions under which a non-negative integral valued stochastic process is an inhomogeneous composed Poisson process. It follows from his results, on account of Jánossy, Rényi and Aczél (1950), that the space process at the time origin is an inhomogeneous Poisson process under the following conditions:

(C<sub>2</sub>):  $\lim_{|I| \rightarrow 0} x(I) = 0$  in probability, for arbitrary  $I \in B(R)$ ;

(C<sub>3</sub>): for disjoint sets  $I_i \in B(R)$  and disjoint set  $K_j \in B(V)$ ,  $x(I_i, K_j)$  are stochastically independent of each other;

(C<sub>4</sub>):  $\lim_{|I| \rightarrow 0} \{P(x(I, v) = 1) / P(x(I, v) \geq 1)\} = 1$ , for arbitrary  $I \in B(R)$  and arbitrary  $v \in V$ .

That is, for  $I \in B(R)$ ,

$$(5.1) \quad q(z; x(I)) = (z - 1) \int_I \lambda(r) dr$$

and, from the fact that  $x(I, v) \leq x(I)$  a.s. for arbitrary

$v \in V$ ,

$$(5.2) \quad q(z; x(I, v)) = (z - 1) \int_I \lambda(r, v) dr,$$

where  $\lambda(r)$  and  $\lambda(r, v)$  are non-negative valued Lebesgue integrable functions such that  $\lambda(r, v) \leq \lambda(r)$ . With little loss of generality, it is assumed that  $\lambda(r)$  and  $\lambda(r, v)$  are  $B(R)$  measurable. It is obvious that  $\underline{E}(x(I))$  and  $\underline{E}(x(I, v))$  are absolutely continuous with respect to Lebesgue measure on  $R$ . Therefore, the infinitesimal s.v.d.f.  $F(v, r)$  exists a.e. on  $R$ , and

$$\lambda(r, v) = F(v, r) \lambda(r) \text{ a.e. } r.$$

Hence, it follows from Condition  $(C_3)$ , Equation (5.2) and the property of generating functions (see Feller (1957), page 251) that

$$(5.3) \quad q(z; x(I, K)) = (z-1) \int_I \int_K dF(v, r) \lambda(r) dr.$$

Moreover, suppose that, on account of Loève ((1963), page 178), an infinitesimal s.v.d.f.  $F(v, r)$  satisfies the following condition:

$(C_5)$ : the infinitesimal s.v.d.f.  $F(v, r)$  is not singular with respect to Lebesgue measure on  $V$ , its absolutely continuous part has the  $B(V) \times B(R)$  measurable derivative  $f(v, r)$  and its purely discontinuous part is composed of  $B(R)$  measurable

jumps  $\{f^d(v_i, r)\}$  on a countable set  $\{v_i\}$  independent of  $r$ .

An infinitesimal s.v.d.f. satisfying Condition  $(C_5)$  is called regular and represented as follows:

$$F(v, r) = \int_0^v f(v, r) dv + \sum_{v_i \in [0, v]} f^d(v_i, r).$$

Therefore, from Equation (5.3) and Fubini's theorem (see Loève (1963), page 136),

$$(5.4) \quad q(z; x(I, K)) = (z-1) \left\{ \int_K \int_I f(v, r) \lambda(r) dr dv + \sum_{v_i \in K} \int_I f^d(v_i, r) \lambda(r) dr \right\}.$$

Lemma 5. Suppose that Conditions  $(C_2)$  to  $(C_5)$  are satisfied. Then  $(C_1; K, h(v))$  is satisfied for arbitrary  $K \in B(V)$  and arbitrary  $h(v) \in C(K)$  whose range belongs to  $R$ . Therefore, for arbitrary  $K \in B(V)$  and arbitrary  $h(v)$ ,  $k(v) \in C(K)$ ,  $S([h(v), k(v)], K)$  is well-defined and

$$(5.5) \quad q(z; S([h(v), k(v)], K)) = (z-1) \left\{ \int_K \int_{h(v)}^{k(v)} f(v, r) \lambda(r) dr dv + \sum_{v_i \in K} \int_{h(v_i)}^{k(v_i)} f^d(v_i, r) \lambda(r) dr \right\}.$$

Proof.

It suffices to prove that for all  $|z| < 1$ , arbitrary  $K \in B(V)$  and arbitrary  $h(v) \in C(K)$ ,

$$\lim_{n \rightarrow \infty} q \left( z; \sum_{i=1}^{k_n} x([\tilde{h}_{ni}, \tilde{h}_{ni}], K_{ni}) \right) = 0.$$

Let

$$I_{ni} = [\tilde{h}_{ni}, \hat{h}_{ni}), e_n(v) = \int_{I_{ni}} f(v, r) \lambda(r) dr \text{ for } v \in K_{ni},$$

$$e_n(v_j) = \int_{I_{ni}} f^d(v_j, r) \lambda(r) dr \text{ for } v_j \in K_{ni}$$

and

$$E_n = \int_K e_n(v) dv + \sum_{v_j \in K} e_n(v_j).$$

It follows from (5.4) and Condition (C<sub>3</sub>) that

$$q \left( z: \sum_{i=1}^{k_n} x([\tilde{h}_{ni}, \hat{h}_{ni}), K_{ni}) \right) = (z-1)E_n.$$

Since  $f(v, r) \lambda(r)$  is integrable on  $I \times V$  for  $I \in B(R)$  such that  $|I| < \infty$ ,  $e_n(v)$  is integrable on  $K$ . Moreover, for arbitrary  $v \in K$ , there exists the sequence of intervals  $\{K_{ni}\}$  such that  $K_{ni} \supset K_{n+1, j} \supset \dots \rightarrow v$  and  $|I_{ni}| \rightarrow 0$ , as  $n \rightarrow \infty$ . Therefore, as  $n \rightarrow \infty$ ,  $e_n(v) \rightarrow 0$  for a.e.  $v$  and  $e_n(v_j) \rightarrow 0$ . It follows from the dominated convergence theorem (see Loève (1963), page 125) that

$$\lim_{n \rightarrow \infty} E_n = \int_K \lim_{n \rightarrow \infty} e_n(v) dv + \sum_{v_j \in K} \lim_{n \rightarrow \infty} e_n(v_j) = 0.$$

The latter half is immediately derived from Lemma 1, (5.4) and condition (C<sub>3</sub>). The proof is concluded.

Similarly, the following Equation (5.6) and Lemma 6

for the time process at the space 'origin is proved under the following Conditions  $(C'_2)$  to  $(C'_5)$ :

- $(C'_2)$ :  $\lim_{|J| \rightarrow 0} y(J) = 0$  in probability, for arbitrary  $J \in B(T)$ ;
- $(C'_3)$ : for disjoint sets  $J_i \in B(T)$  and disjoint sets  $K_k \in B(V)$ ,  $y(J_i, K_k)$  are stochastically independent of each other, and so are  $x(I_i, K_j)$  and  $y(J_i, K_k)$ ;
- $(C'_4)$ :  $\lim_{|J| \rightarrow 0} \{P(y(J, v) = 1) / P(y(J, v) \geq 1)\} = 1$ , for arbitrary  $J \in B(T)$  and arbitrary  $v \in V$ ;
- $(C'_5)$ : the infinitesimal t.v.d.f.  $G(v, t)$  is not singular with respect to Lebesgue measure on  $V$ , its absolutely continuous part has the  $B(V) \times B(T)$  measurable derivative  $g(v, t)$  and its purely discontinuous part is composed of  $B(T)$  measurable jumps  $g^d(v_i, t)$  on a countable set  $\{v_i\}$  independent of  $t$ .

$$(5.6) \quad q(z; y(J, K)) = (z-1) \left\{ \int_K \int_J g(v, t) \rho(t) dt dv + \sum_{v_i \in K} \int_J g^d(v_i, t) \rho(t) dt \right\},$$

where  $\rho(t)$  is a  $B(T)$  measurable non-negative valued integrable function.

Lemma 6. Suppose that Conditions  $(C'_2)$  to  $(C'_5)$  are satisfied. Then  $(C'_1; K, h(v))$  is satisfied for arbitrary

$K$  and arbitrary  $h(v) \in C(K)$  whose range belongs to  $T$ . Therefore, for arbitrary  $K \in B(V)$  and arbitrary  $h(v)$ ,  $k(v) \in C(K)$ ,  $U((h(v), k(v)), K)$  is well-defined and

$$(5.7) \quad q(z; U((h(v), k(v)), K)) = (z-1) \left\{ \int_K \int_{h(v)}^{k(v)} g(v, t) \rho(t) dt dv + \sum_{v_i \in K} \int_{h(v_i)}^{k(v_i)} g^d(v_i, t) \rho(t) dt \right\}.$$

In the remainder of this section, to avoid formal intricacies it is assumed that the infinitesimal s.v.d.f. at the time origin and the infinitesimal t.v.d.f. at the space origin are absolutely continuous with respect to Lebesgue measure on  $V$ . It follows from Condition  $(C'_3)$  that  $S([h(v), k(v)), K)$  and  $U((h'(v), k'(v)), K')$  are stochastically independent of each other. Therefore, combination of Theorems 4 and 6, and Lemmas 5 and 6 implies that the inhomogeneous Poisson initial distributions completely determine distributions of space processes, time processes and observation processes.

Theorem 8. Suppose that Conditions  $(C_2)$  to  $(C_5)$  and  $(C'_2)$  to  $(C'_5)$  are satisfied. Then the initial distributions are inhomogeneous Poisson distributions defined by (5.4) and (5.6). Moreover, under Condition  $(C_0)$  and the absolute continuity of the velocity distribution functions, for  $I = [r, s) \in B(R)$ ,  $J = (t, u] \in B(T)$ ,  $K \in B(V)$ ,  $v^* \in (-\infty, \infty)$

and  $r_0$  such that  $(r_0 + v^*t), (r_0 + v^*u) \in \mathbb{R}$ ,

$$\begin{aligned}
 q(z; x_t(I, K)) &= (z-1) \left\{ \int_{K \cap K_{1x}} \int_{r-tv}^{s-tv} f(v, r) \lambda(r) dr dv \right. \\
 (5.8) \quad &+ \int_{K \cap K_{1x}^c \cap K_{2x}^c} \int_0^{s-tv} f(v, r) \lambda(r) dr dv \\
 &+ \int_{K \cap K_{1x} \cap K_{2x}^c} \int_0^{t-r/v} g(v, t) \rho(t) dt dv \\
 &\left. + \int_{K \cap K_{2x}} \int_{t-s/v}^{t-r/v} g(v, t) \rho(t) dt dv \right\},
 \end{aligned}$$

$$\begin{aligned}
 q(z; y_r(J, K)) &= (z-1) \left\{ \int_{K \cap K_{1y}} \int_{r-uv}^{r-tv} f(v, r) \lambda(r) dr dv \right. \\
 (5.9) \quad &+ \int_{K \cap K_{1y}^c \cap K_{1x}} \int_0^{r-tv} f(v, r) \lambda(r) dr dv \\
 &+ \int_{K \cap K_{1y}^c \cap K_{1x}} \int_0^{u-r/v} g(v, t) \rho(t) dt dv \\
 &\left. + \int_{K \cap K_{1x}^c} \int_{t-r/v}^{u-r/v} g(v, t) \rho(t) dt dv \right\},
 \end{aligned}$$

and

$$\begin{aligned}
 q(z; X(J, K; v^*, r_0)) &= (z-1) \left\{ \int_{K \cap K_{1x} \cap K^*} \int_{r_0+(v^*-v)t}^{r_0+(v^*-v)u} f(v, r) \lambda(r) dr dv \right. \\
 &+ \int_{K \cap K_{1x}^c \cap K_{2x}^c \cap K^*} \int_0^{r_0+(v^*-v)u} f(v, r) \lambda(r) dr dv \\
 (5.10) \quad &+ \int_{K \cap K_{1x}^c \cap K_{2x}^c \cap K^*} \int_0^{t-(r_0+v^*t)/v} g(v, t) \rho(t) dt dv
 \end{aligned}$$



$$\begin{aligned}
& + \int_{K \cap K_{2X} \cap K^*} \int_{u-(r_0+v^*u)/v}^{t-(r_0+v^*t)/v} g(v,t) \rho(t) dt dv \\
& + \int_{K \cap K_{2X}^c \cap (K^*)^c} \int_{r_0-(v-v^*)u}^{r_0-(v-v^*)t} f(v,r) \lambda(r) dr dv \\
& + \int_{K \cap K_{1X} \cap K_{2X} \cap (K^*)^c} \int_0^{r_0-(v-v^*)t} f(v,r) \lambda(r) dr dv \\
& + \int_{K \cap K_{1X} \cap K_{2X} \cap (K^*)^c} \int_0^{u-(r_0+v^*u)/v} g(v,t) \rho(t) dt dv \\
& + \int_{K \cap K_{1X}^c \cap (K^*)^c} \int_{t-(r_0+v^*t)/v}^{u-(r_0+v^*u)/v} g(v,t) \rho(t) dt dv ,
\end{aligned}$$

where  $K_{1X} = [0, r/t]$ ,  $K_{2X} = (s/t, \infty)$ ,  $K_{1Y} = [0, r/u]$ ,  
 $K_{1X} = [0, v^* + r_0/t]$ ,  $K_{2X} = (v^* + r_0/u, \infty)$  and  $K^* = [0, v^*]$ .

This theorem shows that for the inhomogeneous Poisson initial distributions the space process at an arbitrary time, the time process at an arbitrary point and observation processes are also inhomogeneous Poisson processes. Moreover, it is clear that velocity distribution functions defined by  $(D_2)$  or  $(D'_2)$  have property (E) or  $(E')$ . If  $x(I, K)$  or  $y(J, K)$  is defined respectively for  $R = (-\infty, R']$ ,  $I \in B(R)$   $K \in B(V)$ , or for  $T = (-\infty, T']$ ,  $J \in B(T)$  and  $K \in B(V)$ , then the above theorem is reduced to the following corollaries from the condition that  $\lambda, \rho, f$  and  $g$  are Borel measurable.

Corollary 1. For  $R = (-\infty, R')$ ,  $I = [r, s) \in B(R)$ ,  $J = (t, u] \in B(T)$  and  $K \in B(V)$ ,

$$q(z; x_t(I, K)) = (z-1) \int_I \int_K f(v, r-tv) \lambda(r-tv) dv dr,$$

$$q(z; y_r(J, K)) = (z-1) \int_J \int_K v f(v, r-tv) \lambda(r-tv) dv dt,$$

and for  $v^* \in (-\infty, \infty)$  and  $r_0$  such that  $(r_0 + v^*t), (r_0 + v^*u) \in R$ ,

$$q(z; X(J, K; v^*, r_0)) =$$

$$(z-1) \left\{ \int_J \int_{K \cap K^*} (v^* - v) f(v, r_0 + (v^* - v)t) \lambda(r_0 + (v^* - v)t) dv dt \right. \\ \left. + \int_J \int_{K \setminus (K^*)^c} (v - v^*) f(v, r_0 - (v - v^*)t) \lambda(r_0 - (v - v^*)t) dv dt \right\}$$

where  $K^* = [0, v^*]$ .

Corollary 2. Let  $T = (-\infty, T']$ ,  $I = [r, s) \in B(R)$ ,  $J = (t, u] \in B(T)$  and  $K \in B(V)$ . Suppose that  $y(J, K') = 0$  a.s., for arbitrary  $J \in B(T)$  and  $K' = [0, \varepsilon)$ , where  $\varepsilon$  is a small positive number. Then,

$$q(z; x_t(I, K)) = (z-1) \int_I \int_K g(v, t-r/v) \rho(t-r/v)/v dv dr,$$

$$q(z; y_r(J, K)) = (z-1) \int_J \int_K g(v, t-r/v) \rho(t-r/v) dv dt$$

and for  $v^* \in (-\infty, \infty)$  and  $r_0$  such that  $(r_0 + v^*t), (r_0 + v^*u) \in R$ ,

$$q(z; X(J, K; v^*, r_0)) =$$

$$(z-1) \left\{ \int_J \int_{K \cap K^*} (v^*/v-1) g(v, t-(r_0+v^*t)/v) \rho(t-(r_0+v^*t)/v) dv dt \right. \\ \left. + \int_J \int_{K \cap (K^*)^c} (1-v^*/v) g(v, t-(r_0+v^*t)/v) \rho(t-(r_0+v^*t)/v) dv dt \right\},$$

where  $K^* = [0, v^*]$ .

Since  $x_t(I, K)$  and  $y_r(J, K)$  are inhomogeneous Poisson processes, it is legitimate to put

$$q(z; x_t(I, K)) = (z-1) \int_I \int_K f_t(v, r) \lambda_t(r) dv dr$$

and

$$q(z; y_r(J, K)) = (z-1) \int_J \int_K g_r(v, t) \rho_r(t) dv dt.$$

Hence, it follows from Corollaries 1 and 2 that

$$(5.11) \quad \begin{aligned} \lambda_t(r) &= \int_v f(v, r-tv) \lambda(r-tv) dv \\ &= \int_v g(v, t-r/v) \rho(t-r/v) v dv, \text{ for a.e. } r, \end{aligned}$$

$$(5.12) \quad \begin{aligned} f_t(v, r) \lambda_t(r) &= f(v, r-tv) \lambda(r-tv) \\ &= g(v, t-r/v) \rho(t-r/v)/v, \text{ for a.e. } r \text{ and } v, \end{aligned}$$

$$(5.13) \quad \begin{aligned} \rho_r(t) &= \int_v v f(v, r-tv) \lambda(r-tv) dv \\ &= \int_v g(v, t-r/v) \rho(t-r/v) dv, \text{ for a.e. } t, \end{aligned}$$

and

$$\begin{aligned}
 (5.14) \quad g_r(v, t) \rho_r(t) &= v f(v, r - tv) \lambda(r - tv) \\
 &= g(v, t - r/v) \rho(t - r/v), \text{ for a.e.} \\
 &\quad \text{and } v.
 \end{aligned}$$

The following question arises necessarily: the observer starts from  $r$  at  $t$  with velocity  $v^* \in (-\infty, \infty)$ . What velocity  $v^*$  minimizes or maximizes (if possible) the expectation of the number of vehicles observed during the time interval  $J = (t, u]$ ? That is, what velocity  $\tilde{v}^*$  or  $\hat{v}^*$  attains min or max  $E(X(J, V; v^*, r - tv^*))$  respectively? It follows from (2.17) and (5.5) that

$$\begin{aligned}
 E(X(J, V; v^*, r - tv^*)) &= \int_{K^*} \int_{r - tv}^{r + v^* |J| - vu} f(v, r) \lambda(r) dr dv \\
 &\quad + \int_{(K^*)^c} \int_{r + v^* |J| - vu}^{r - tv} f(v, r) \lambda(r) dr dv.
 \end{aligned}$$

Suppose that  $\lambda(r)$  and  $f(v, r)$  are continuous in  $r$ .

Therefore,

$$\begin{aligned}
 \partial E(X(J, V; v^*, r - tv^*)) / \partial v^* &= |J| \left\{ \int_0^{v^*} f(v, r + v^* |J| - vu) \lambda(r + v^* |J| - vu) dv \right. \\
 &\quad \left. - \int_{v^*}^{\infty} f(v, r + v^* |J| - vu) \lambda(r + v^* |J| - vu) dv \right\} \\
 &= |J| \lambda_u(r + v^* |J|) \left\{ \int_0^{v^*} f_u(v, r + v^* |J|) dv \right.
 \end{aligned}$$

$$- \int_{v^*}^{\infty} f_u(v, r+v^*|J|) dv \},$$

where

$$(5.15) \quad \lambda_u(r+v^*|J|) = \int_V f(v, r+v^*|J|-vu) \lambda(r+v^*|J|-vu) dv$$

and

$$(5.16) \quad f_u(v, r+v^*|J|) \lambda_u(r+v^*|J|) = f(v, r+v^*|J|-vu) \lambda(r+v^*|J|-vu).$$

It is clear that  $\partial E(X)/\partial v^*$  is continuous in  $v^*$ , non-positive for non-positive  $v^*$  and non-negative for sufficiently large  $v^*$ . Therefore, if

$$\begin{aligned} \lim_{v^* \rightarrow -\infty} E(X(J, V; v^*, r-tv^*)) &\equiv E(X^-(J, V; \infty, \cdot)) \\ &= \int_V \int_{-\infty}^{r-tv} f(v, r) \lambda(r) dr dv < \infty, \end{aligned}$$

and

$$\begin{aligned} \lim_{v^* \rightarrow \infty} E(X(J, V; v^*, r-tv^*)) &\equiv E(X^+(J, V; \infty, \cdot)) \\ &= \int_V \int_{r-tv}^{\infty} f(v, r) \lambda(r) dr dv < \infty, \end{aligned}$$

then

$$\sup_{v^*} E(X(J, V; v^*, r-tv^*)) = \max \{E(X^-(J, V; \infty, \cdot)), E(X^+(J, V; \infty, \cdot))\}.$$

Moreover, the necessary condition for  $\tilde{v}^*$  to minimize  $E(X(J, V; v^*, r - tv^*))$  is:

$$(5.17) \quad \lambda_u(r + v^* | J) \left\{ \int_0^{\tilde{v}^*} f_u(v, r + v^* | J) dv - \int_{\tilde{v}^*}^{\infty} f_u(v, r + v^* | J) dv \right\} = 0;$$

there exists at least one value  $\tilde{v}^*$  satisfying (5.17).

Finally, the results obtained above are applied to a homogeneous Poisson traffic flow. Suppose that  $\lambda(r) = \lambda$  and  $f(v, r) = f(v)$ , where  $\lambda$  is a positive number and  $f(v)$  is an ordinary density function. It follows from (5.11) and (5.12) that  $\lambda_t(r) = \lambda$  and  $f_t(v, r) = f(v)$ . Moreover, from (5.13) and (5.14),

$$(5.18) \quad \rho_r(t) = \lambda \int_V v f(v) dv \equiv \rho \quad \text{and} \quad g_r(v, t) = v f(v) / \int_V v f(v) dv \equiv g(v).$$

Equation (5.18) is familiar in road traffic flow theory (see Haight (1963), page 116). Furthermore, for Corollaries 1 and 2,

$$(5.19) \quad q(z; X(J, K; v^*, r_0)) = (z-1)\lambda |J| \left\{ \int_{K \cap K^*} (v^* - v) f(v) dv + \int_{K \cap (K^*)^c} (v - v^*) f(v) dv \right\}$$

$$(5.20) \quad = (z-1)\rho |J| \left\{ \int_{K \cap K^*} (v^*/v - 1) g(v) dv \right.$$

$$+ \int_{K_n(K^*)} c^{(1-v^*/v)} g(v) dv \}$$

and from Equations (5.15), (5.16) and (5.17),

$$(5.21) \quad \int_0^{\tilde{v}^*} f(v) dv - \int_{\tilde{v}^*}^{\infty} f(v) dv = 0;$$

that is,  $\tilde{v}^*$  is the median of the s.v.d.f. Equation (5.20) agrees with the result obtained by Rényi (1964) and (5.19) and (5.21) agree with the results obtained by Weiss and Herman (1962), though they deal with only  $x^+$ .

### 3. 6. Concluding remarks

Throughout this chapter,  $V$  is assumed to be  $[0, \infty)$ . If  $V = (-\infty, \infty)$ , then generalized traffic flow including also the traffic flow on the opposite lane can be dealt with in the same manner as that of this chapter. This gives the work by Wardrop and Charlesworth (1954) a theoretical basis. Besides,  $v^*$  is assumed to be constant. This assumption also is not essential. In case that  $v^*$  varies with  $t$ , put  $\bar{v}^* = \int_J v^*(t) dt / |J|$ . Then results obtained for  $X$  remain valid with  $v^*$  replaced by  $\bar{v}^*$ . It is to be noted, however, that in this case  $X^+$  does not contain the number of vehicles which the observer once overtakes and by which he is afterwards overtaken.

Conditions  $(C_4)$  and  $(C'_4)$  in Section 3. 5 are not essential. If these conditions are dropped, then the initial distributions become inhomogeneous composed Poisson distributions, as shown in the preceding chapter. Therefore all results obtained in the preceding chapter can be derived from Theorems 4 and 6 in this chapter.

All results obtained in this chapter apply to one-dimensional inhomogeneous flows of particles with their own constant velocities. Another interesting example of such flows is a flow of  $\mu$ -mesons in cosmic particles. In observations on this flow, effects of a velocity of an observer (for example, a rocket or an artificial



satellite), which is shown in Theorem 6, may be an interesting problem from a theoretical point of view.

The possible modification to the present discussion would involve some form of interaction between vehicles. Such modification, however, might be too complicated to analyze.

## Chapter 4. Traffic Light Queues with Independent Arrivals

### 4. 1. Introduction

Many researchers have investigated queues of vehicles at intersections controlled by traffic lights, or traffic light queues. A traffic light is called fixed-cycle, if it has a cycle of fixed length. It is called vehicle-actuated, if it contains some type of detectors which gather information about events occurring in the intersecting traffic streams. In particular, it is called semi-vehicle-actuated, if it contains detectors on the only one road of the two intersecting ones that is usually called the minor road. Thus, traffic lights are classified according to their functions into fixed-cycle, vehicle-actuated and semi-vehicle-actuated traffic lights.

For a fixed-cycle traffic light queue, Beckmann, McGuire and Winsten (1956) proposed a discrete time queueing model with binomial arrivals and regular departure headways and derived a relation between the stationary mean delay per vehicle and the stationary mean queue-length at the beginning of a red period of the traffic light. Newell(1960a) also dealt with the same

model and obtained the probability generating function of the stationary queue-length distribution. This model has been investigated by Dunne (1967), Potts (1967) and Smit (1971). Webster (1958), Hight (1959) and Buckley and Wheeler (1964) considered models with Poisson arrivals and regular departure headways and investigated certain properties of the queue-length. Darroch (1964) discussed a discrete time model with stationary and independent arrivals and regular departure headways and derived a necessary and sufficient condition for the stationary queue-length distribution to exist and its probability generating function. The above authors, Little (1961), Miller (1963b), Kleinecke (1964), Newell (1965), McNeil (1968) and Siskind (1970) gave approximate expressions of the stationary mean delay per vehicle for fixed-cycle traffic light queues of various types. Related problems have been discussed by Newell (1956, 1959a), Uematu (1957), Smeed (1957), Gazis, Herman and Maradudin (1960), Olson and Rothery (1961), Dick (1964), Webster (1964), Anker and Gafarian (1967), Blunden and Pretty (1967), Anker, Gafarian and Gray (1968), Huddart (1969), Thedéen (1969 a) and Dickey and Montgomery (1970). On the other hand, Garwood (1940), Darroch, Newell and Morris (1964), Newell (1969) and Newell and Osuna (1969) investigated vehicle-actuated traffic light queues.

Haight (1959) and Little (1971) discussed semi-vehicle-actuated traffic light queues. All of the researchers mentioned above dealt with the queue-length.

In this chapter, however, the total remaining departure headway of vehicles in the queue is dealt with, which corresponds to the "server occupation time" used in queueing theory if departure headways are considered as service times. It can be shown in the next section that a fixed-cycle traffic light queue with rather general arrivals and departure headways is reduced to a generalized model of the GI/G/1 queueing process originated by Lindley (1952). In Section 4. 3, this generalized model is discussed. A necessary and sufficient condition is derived under which a limiting distribution exists, and an integral equation is obtained which the limiting distribution function satisfies. The above results lead directly to a necessary and sufficient condition for a stationary distribution of the fixed-cycle traffic light queue to exist. This condition includes Darroach's one (1964) as a simple case. It should be noted that this condition remains valid for a vehicle-actuated traffic light queue, because vehicle-actuated traffic lights operate on a fixed cycle under heavy traffic demands (see Webster and Cobbe (1966)). In Section 4. 4, a successive approximation method of the

limiting distribution function is presented. In Section 4. 5, it can be shown that a semi-vehicle-actuated traffic light queue with stationary and independent arrivals and general departure headways is also reduced to the generalized model discussed in Section 4. 3. A necessary and sufficient condition is derived under which the semi-vehicle-actuated traffic light queue has a stationary distribution. Finally, two typical examples of semi-vehicle-actuated traffic light queues are discussed.

#### 4. 2. Fixed-cycle traffic light queue

Consider a traffic into an intersection controlled by a fixed-cycle traffic light. It can be assumed without loss of generality that one cycle of the traffic light comprises one red period and one green period, since the amber period can be thought of as effectively red or effectively green. Moreover, as is usually done, those vehicles which go straight on or turn left and those which turn right are separately discussed. To begin with, let be concerned with only vehicles which go straight on or turn left. The model discussed in this chapter satisfies the five conditions (A) through (E) mentioned below.

- (A) One cycle of the traffic light is composed of one red period of fixed length  $r$  and one successive green period of fixed length  $g$ .

Therefore the cycle length of the traffic light is  $T (=r+g)$ . It is supposed that the  $n$ th cycle of the traffic light begins at time  $nT$  and finishes at time  $(n+1)T$ , where  $n=0,1,2,\dots$ . Denote by  $x_n(t)$  ( $0 \leq t \leq T$ ) the number of vehicles which arrive at the intersection during the time interval of length  $t$  from the beginning of the  $n$ th cycle, i.e. the time interval  $(nT, nT+t]$ . Vehicles arriving at the intersection in the red period have to wait until the beginning of the green period at

the shortest. Therefore it suffices for the following analysis to specify stochastic properties of the arrival process in the  $n$ th green period  $x_n(r+t)$  ( $0 \leq t \leq g$ ). Arrival processes in green periods  $x_n(r+t)$  ( $0 \leq t \leq g$ ) are called mutually independent, if for arbitrary integers  $k_n$  and sequences  $\{t_{ni}; i=1,2,\dots,k_n\}$  such that  $0 \leq t_{n1} \leq \dots \leq t_{nk_n} \leq g$ , random vectors  $(x_n(r), x_n(r+t_{n1}), \dots, x_n(r+t_{nk_n}), x_n(T))$  are mutually independent. Moreover, arrival processes in green periods are called identically distributed, if for any integer  $k$  and sequence  $\{t_i; i=1,2,\dots,k\}$  such that  $0 \leq t_1 \leq \dots \leq t_k \leq g$ , random vectors  $(x_n(r), x_n(r+t_1), \dots, x_n(r+t_k), x_n(T))$  are identically distributed. Let  $\underline{E}(\cdot)$  denote the expectation of  $\cdot$ . Suppose that arrival processes in green periods satisfy the following condition:

(B) The arrival processes in green periods  $x_n(r+t)$  ( $0 \leq t \leq g$ ) are mutually independent and identically distributed and  $\underline{E}\{x_n(T)\} = \lambda T (< \infty)$ .

This condition requires none of stationarity, absence of after-effects and orderliness of the stream of vehicles arriving at the intersection (see Khintchine (1960)). Conversely, if the stream of arriving vehicles is a stationary one without after-effects and has finite intensity  $\lambda$ , the arrival processes in green periods satisfy Condition (B). Thus, Condition (B) seems to be

rather practical and reasonable. Suppose that some vehicles wait in the queue at the end of the red period. As the light turns green, the  $i$ th ( $i=1,2,\dots$ ) vehicle in the queue crosses the stop line at time  $t_i$  ( $0 < t_i \leq g$ ) from the beginning of the green period and passes through the intersection. The time interval between  $t_{i-1}$  and  $t_i$  ( $i=2,3,\dots$ ) is called the departure headway of the  $i$ th vehicle. However, the time interval  $t_1$  is much longer than the departure headway of the  $i$ th vehicle. Because the time interval  $t_1$  is composed of the starting delay for getting entire queue into motion and the departure headway of the first vehicle (see Drew (1968), pages 104 and 138). In the present and the next chapters, for vehicles going straight on or turning left, departure headways and starting delays are called service times and lost times (see Miller (1963)), respectively.

- (C) Service times of arriving vehicles and lost times in all cycles are mutually independent and identically distributed random variables with finite mean respectively, independent each other and independent of the arrival process  $x_n(t)$ .

This condition seems to be reasonable and practical.

Denote by  $s_i$  the service time of the  $i$ th vehicle arriving in the  $n$ th cycle and by  $K_n(r+t)$  the total service time of vehicles arriving in the time interval  $r+t$  from the



beginning of the  $n$ th cycle. Clearly, the following relation holds:

$$K_n(r+t) = \sum_{i=1}^{x_n(r+t)} s_i .$$

Therefore it follows from Conditions (B) and (C) that for fixed  $t(0 \leq t \leq g)$ ,  $\{K_n(r+t)\}$  is a sequence of mutually independent and identically distributed random variables with finite mean. Further, without loss of generality,  $K_n(r+t)$  can be assumed separable (see Doob (1953), page 57).

(D) Once the queue discharges, it does not reform during the remaining green period.

That is, those vehicles which arrive during the green period and find the queue empty, pass through the intersection without delay. This condition represents the most distinctive feature of the traffic light queue.

(E) The service discipline is "first-come, first-served" and "preemptive resume."

The former condition is frequently made and it has been implicitly done in the explanation of the departure headway and the starting delay. The latter one means that if the light turns red before a vehicle at the head of the queue crosses the stop line, then the vehicle stops and crosses the stop line at the time equal to its remaining service time plus the lost time from the

beginning of the next green period. Though it may be more profitable (in particular, for right turning vehicles discussed later) to assume "preemptive repeat identical", it seems to be difficult to investigate the model with this service discipline. It is to be noted, however, that in case of the model used frequently which has regular service times, regular lost times and effective green periods of the integral times length of the service time, the above model is identical with the one with "preemptive resume". Denote by  $w_n$  ( $n=0,1,2,\dots$ ) the total remaining service time of the vehicles in the queue, that is, the server occupation time at the beginning of the  $n$ th cycle. In particular,  $w_0$  is called the initial server occupation time. Let  $\ell_n$  be the lost time in the  $n$ th cycle. Clearly  $\ell_n \leq g$  almost surely (a.s.). It follows from Conditions (A) and (D) that  $w_{n+1}$  vanishes if the server occupation time at the end of the  $n$ th red period is equal to zero or if the server occupation time vanishes during the  $n$ th green period. Therefore if  $w_n + K_n(r) > 0$  and  $\inf_{0 \leq t \leq g} \{w_n + K_n(r+t) + \ell_n - t\} > 0$ , then by Condition (E),  $w_{n+1}$  takes the value  $\{w_n + K_n(T) - (g - \ell_n)\}$ . Hence the recurrence relation between  $w_n$  and  $w_{n+1}$  ( $n=0,1,2,\dots$ ) holds as follows:

$$(2.1) \quad w_{n+1} = w_n + K_n(T) - (g - \ell_n),$$

$$\begin{aligned} & \text{if } w_n + \inf\{K_n(r), \inf_{0 < t \leq g} \{K_n(r+t) + l_n - t\}\} > 0, \\ & = 0 \quad , \text{ otherwise.} \end{aligned}$$

It should be noted that  $\inf\{K_n(r), \inf_{0 < t \leq g} \{K_n(r+t) + l_n - t\}\}$  is a well-defined random variable, since  $K_n(r+t)$  is separable.

Let be concerned with only vehicles turning right. The characteristic feature of these vehicles is that the vehicle at the head of the queue is hindered from turning right by both the red signal and opposing vehicles during the green period. The time from the beginning of the green period to the instant when the first vehicle of the queue reaches the position to turn right is called the starting delay for vehicles turning right. It is supposed that the vehicle in the position to turn right, which is at the head of the queue, can not move during the time interval of the specified length before the passage of the opposing vehicle. Then the opposing traffic can be thought of as an alternating succession of periods during which moving is impossible and periods during which moving is possible. The former periods are called blocks. Moreover the starting delay and blocks are collectively called the lost time and the time which the vehicle in the position to turn right takes to cross the opposite lane is called the service time for vehicles turning

right. The same Conditions (A) through (E) as for vehicles going straight on imply that the same recurrence relation as (2. 1) holds also for vehicles turning right. It may be noted, however, that Conditions (D) and (E) roughly approximate to the actual behavior of vehicles turning right. Therefore the fixed-cycle traffic light queue is reduced to model (2. 1).

Put

$$(2. 2) \quad u_n = K_n(T) - (g - l_n)$$

$$(2. 3) \text{ and } v_n = \inf\{K_n(r), \inf_{0 \leq t \leq g} \{K_n(r+t) + l_n - t\}\}.$$

Since for fixed  $t (0 \leq t \leq g)$ ,  $\{K_n(r+t)\}$  is a sequence of mutually independent and identically distributed random variables with finite mean and so is  $\{l_n\}$ ,  $\{u_n\}$  and  $\{v_n\}$  are sequences of mutually independent and identically distributed random variables with finite mean. Clearly for  $n=0, 1, 2, \dots$ ,

$$(2. 4) \quad u_n \geq v_n \quad \text{a.s.}$$

$$\text{Let } U(x) = \Pr\{u_n \leq x\}, \quad V(x) = \Pr\{v_n \leq x\},$$

$$\text{and } V(x | y) = \Pr\{v_n \leq x \mid u_n = y\}.$$

By (2. 3),  $v_n$  is a.s. bounded to the below; that is, there exists a finite positive number  $c$  such that for an arbitrary positive number  $\epsilon$ ,

$$(2.5) \quad V(-c-0)=0 \quad \text{and} \quad V(-c+\varepsilon) > 0.$$

Moreover, by (2.4) and (2.5),  $E(|u_n|) < \infty$ . Hence, model (2.1) or the fixed-cycle traffic light queue is finally reduced to the following rather general model: for  $n=0,1,2,\dots$ ,

$$(2.6) \quad \begin{aligned} w_{n+1} &= w_n + u_n, & \text{if } w_n + v_n > 0, \\ &= 0, & \text{otherwise,} \end{aligned}$$

where  $\{u_n\}$  and  $\{v_n\}$  are sequences of mutually independent and identically distributed random variables with finite mean. In the next section, this generalized model is discussed which satisfies (2.4) and, if stated, (2.5).

Let  $[x]^+ = \max(0, x)$  and  $[x]^- = \min(0, x)$ .

The above model is a generalized model of the GI/G/1 queueing process discussed by Lindley (1952) which is expressed by  $w_{n+1} = [w_n + u_n]^+$ . The Lindley's model has been extended by the following authors and others: Finch (1959) considered the model expressed by

$$\begin{aligned} w_{n+1} &= w_n + u_n, & \text{if } w_n + u_n > 0, \\ &= v_n, & \text{otherwise,} \end{aligned}$$

where  $u_n$  and  $v_n$  are independent. Daley (1965) discussed a general customer impatience problem. Cohn (1967) discussed the two models:

$$w_{n+1} = K + [w_n + u_n]^+ - K^-$$

and  $w_{n+1} = [v_n + [w_n + u_n]^-]^+$ ,

where  $u_n$  and  $v_n$  are independent, and  $K$  is a positive constant number.

#### 4. 3. The limiting distribution

The model to be discussed is (2. 6) with (2. 4).

Let

$$W_n(x) = \Pr\{w_n \leq x\}$$

and 
$$W_n^0(x) = \Pr\{w_n \leq x \mid w_0 = 0\}.$$

Clearly,

$$W_n(x) = 0 \quad \text{for } x < 0,$$

and 
$$W_0^0(x) = 0 \quad (\text{for } x < 0), \quad = 1 \quad (\text{for } x \geq 0).$$

The distribution function (d.f.)  $W_0(x)$  is called the initial d.f. The purpose of this section is to derive a necessary and sufficient condition under which  $W_n(x)$  converges to an honest limiting d.f. as  $n$  tends to infinity, and an integral equation which the limiting d.f. satisfies. From (2. 6) it follows that for  $x \geq 0$  and  $n=0,1,2,\dots$ ,

$$\begin{aligned} W_{n+1}(x) &= \Pr\{w_n + v_n \leq 0\} + \Pr\{w_n + v_n > 0, w_n + u_n \leq x\} \\ &= \Pr\{w_n + u_n \leq x\} + \Pr\{w_n + v_n \leq 0, w_n + u_n > x\} \\ &= \Pr\{w_n + u_n \leq x, u_n - v_n < x\} + \Pr\{w_n + v_n \leq 0, u_n - v_n \geq x\}. \end{aligned}$$

Consequently, for  $x \geq 0$  and  $n=0,1,2,\dots$ ,

$$\begin{aligned} (3.1) \quad W_{n+1}(x) &= \int_{-\infty}^{x+0} \{W_n(x-y)[1-V(y-x|y)] + \int_{-\infty}^{y-x} W_n(-z) dV(z|y)\} \\ &\quad dU(y) + \int_{x+0}^{\infty} \int_{-\infty}^{+0} W_n(-z) dV(z|y) dU(y). \end{aligned}$$

Lemma 1.

As  $n$  tends to infinity,  $W_n^0(x)$  converges to an honest or dishonest d.f.  $W^0(x)$  and  $W^0(x)$  satisfies the following integral equation:

$$(3.2) \quad W^0(x) = \int_{-\infty}^{x+0} \{W^0(x-y)[1-V(y-x|y)] + \int_{-\infty}^{y-x} W^0(-z)dV(z|y)\} dU(y) \\ + \int_{x+0}^{\infty} \int_{-\infty}^{+0} W^0(-z)dV(z|y)dU(y), \text{ for } x \geq 0, \\ = 0, \text{ for } x < 0.$$

Proof.

Since (3.1) is valid for  $W_n^0(x)$ , for  $n=0$  and  $x \geq 0$ ,

$$W_1^0(x) = U(x) + \int_{x+0}^{\infty} V(0|y)dU(y) \leq W_0^0(x)$$

Suppose that  $W_{n-1}^0(x) \geq W_n^0(x)$ . Then, by (3.1),

$$W_n^0(x) - W_{n+1}^0(x) = \int_{-\infty}^{x+0} \{[W_{n-1}^0(x) - W_n^0(x)][1-V(y-x|y)] \\ + \int_{-\infty}^{y-x} [W_{n-1}^0(-z) - W_n^0(-z)]dV(z|y)\} dU(y) \\ + \int_{x+0}^{\infty} \int_{-\infty}^{+0} [W_{n-1}^0(-z) - W_n^0(-z)]dV(z|y)dU(y) \geq 0.$$

Therefore, by mathematical induction,

$$W_n^0(x) \geq W_{n+1}^0(x), \text{ for } n=0,1,2,\dots$$

Since  $W_n^0(x)$  are nonnegative-valued nondecreasing functions,  $W^0(x) = \lim_{n \rightarrow \infty} W_n^0(x)$  exists uniquely and, by monotone convergence theorem (see Loève (1963), page 124), satisfies Integral Equation (3.2). Clearly,  $W^0(x)$  is



nonnegative-valued and nondecreasing in  $x$ . Moreover,  $W^0(x)$  is continuous from the right. In fact, there exist positive numbers  $n$  and  $\delta$  such that for an arbitrary positive number  $\epsilon$  and  $x \geq 0$ ,

$$W_n^0(x) - W^0(x) < \epsilon/2 \text{ and } W_n^0(x+\delta) - W_n^0(x) < \epsilon/2.$$

Consequently,  $W^0(x+\delta) - W^0(x) \leq W_n^0(x+\delta) - W_n^0(x) + W_n^0(x) - W^0(x) < \epsilon$ ; that is,  $W^0(x)$  is continuous from the right. However,  $W^0(x)$  may be or may not be an honest d.f.

The main problem to solve in the sequel is to decide whether  $W^0(x)$  is an honest d.f. or not.

Lemma 2.

If  $E(u_n) < 0$ , then  $W(x) = \lim_{n \rightarrow \infty} W_n(x)$  exists and is an honest d.f. Further,  $W(x)$  is independent of  $W_0(x)$  and is the unique honest d.f. which satisfies Integral Equation (3. 2) with  $W^0(x)$  exchanged by  $W(x)$ .

Proof.

Consider the GI/G/1 queueing process  $\hat{w}_{n+1} = [\hat{w}_n + u_n]^+$ .

Suppose that  $w_0 = \hat{w}_0 = 0$  a.s. Since  $u_0 \geq v_0$  a.s.,  $\hat{w}_1 \geq w_1$  a.s.

If  $\hat{w}_n \geq w_n$  a.s., then

$$\begin{aligned} \hat{w}_{n+1} - w_{n+1} &= I\{\hat{w}_n + u_n > 0, w_n + v_n > 0\}(\hat{w}_n - w_n) \\ &\quad + I\{\hat{w}_n + u_n > 0, w_n + v_n \leq 0\}(\hat{w}_n + u_n) \geq 0 \text{ a.s.,} \end{aligned}$$

where  $I\{\cdot\}$  denotes the characteristic function of event  $\cdot$ .

Consequently  $\hat{w}_n \geq w_n$  a.s., for  $n=0,1,2,\dots$ . It follows from Lindley (1952) that  $\hat{w}_n$  converges in distribution to

a proper random variable. Hence,  $W^0(x)$  is the honest d.f. Let

$$S_n = \sum_{i=0}^n u_i$$

and  $r_n = \min\{v_0, S_0 + v_1, \dots, S_{n-1} + v_n\}$ .

Denote by  $T$  the first passage time to  $\{w_n = 0\}$ ; that is,

$$T = \{n; w_0 + r_{n-1} > 0 \text{ and } w_0 + r_n \leq 0\}.$$

Let  $p_n = \Pr\{T=n\}$ . Clearly  $w_0 + r_n \leq w_0 + S_n$  a.s. and, by the strong law of large numbers,

$$w_0 + S_n \rightarrow -\infty \text{ a.s., as } n \rightarrow \infty.$$

Therefore,  $T$  is a.s. finite; that is,  $\sum_{n=0}^{\infty} p_n = 1$ . Since

$$W_n(x) = \sum_{i=0}^n p_i W_{n-i}^0(x)$$

and  $\lim_{n \rightarrow \infty} W_{n-i}^0(x) = W^0(x)$ ,

$$W(x) = \lim_{n \rightarrow \infty} W_n(x) = W^0(x);$$

that is  $W(x)$  is independent of  $W_0(x)$  and satisfies Integral Equation (3. 2) with  $W^0(x)$  exchanged by  $W(x)$ . Suppose that this integral equation has another honest d.f.  $W^*(x)$  as solutions. Put  $W_0(x) = W^*(x)$ . Then  $W_n(x) = W^*(x)$  for all  $n=1, 2, \dots$ . Therefore  $W(x) = W^*(x)$ . The proof is concluded.

### Lemma 3.

If  $E(u_n) \geq 0$ , (2. 5) is satisfied and

$$(3.3) \quad p=1-V(0) > 0,$$

then  $W(x)=0$  for  $-\infty < x < \infty$ .

Proof.

Denote by  $w_n^0$  the random variable  $w_n$  conditioned by  $w_0=0$  a.s. Clearly  $w_0 \geq 0$  a.s. and if  $w_n \geq w_n^0$  a.s.,

$$\begin{aligned} \text{then } w_{n+1} - w_{n+1}^0 &= I\{w_n^0 + v_n > 0\}(w_n - w_n^0) + I\{w_n + v_n > 0, w_n^0 + v_n \leq 0\}(w_n + u_n) \\ &\geq 0 \text{ a.s.} \end{aligned}$$

Consequently, by induction,

$$w_n \geq w_n^0 \text{ a.s., for } n=0,1,2,\dots$$

This inequality implies that

$$W_n^0(x) \geq W_n(x), \text{ for all } x \text{ and } n=0,1,2,\dots$$

Hence it suffices to prove that  $W^0(x)=0$  for  $-\infty < x < \infty$ .

$$\text{Let } T^0 = \{n; r_{n-1} > 0, r_n \leq 0\},$$

$$\hat{r}_n = \min\{S_0, S_1, \dots, S_n\},$$

$$\text{and } \hat{T} = \{n; \hat{r}_{n-1} > 0, \hat{r}_n \leq 0\}.$$

Let  $A$  be the event  $\{r_k > 0, S_k > c\}$  for a positive integer  $k$  such that  $\Pr\{u_n > c/k | v_n > 0\} > 0$ ; finite  $k$  exists always by (2.4) and (3.3). Denote by  $\mathcal{E}$  the recurrent event  $\{w_n^0 = 0\}$ . To begin with, the case of  $\underline{E}(u_n) = 0$  is discussed. Since by (2.4),  $\hat{r}_n \geq r_n$  a.s.,  $\hat{T} \geq T^0$  a.s. Note that  $\underline{E}(|u_n|) < \infty$ , by (2.4) and (2.5). According to Spitzer (1956),  $\hat{T}$  is a proper random variable with  $\underline{E}(\hat{T}) = \infty$ .

Therefore,  $T^0$  is also a proper random variable and  $\mathcal{E}$  is a persistent and aperiodic recurrent event. By (2. 5) and (3. 3),

$$\begin{aligned} \underline{E}(T^0) &\geq \int_A T^0 dP = \Pr(A) \underline{E}(T^0|A) \\ &\geq \Pr(v_i > 0, u_i > c/k, i=0,1,2,\dots,k) \{ \underline{E}(\hat{T}) + k \} \\ &= \{ p \cdot \Pr\{u_0 > c/k | v_0 > 0\} \}^k \{ \underline{E}(\hat{T}) + k \} = \infty. \end{aligned}$$

Consequently, from Feller (1957, page 286),  $\lim_{n \rightarrow \infty} \Pr\{w_n^0 = 0\} = 0$ . Since by (3. 2),  $w^0(0) = \int_{-c}^{+0} w^0(-y) dV(y)$ ,  $w^0(0) = 0$  implies  $w^0(c) = 0$ . Moreover, since  $w^0(c) = \int_{-c}^0 w^0(c-y) dU(y)$ ,  $w^0(c) = 0$  implies  $w^0(c+d) = 0$ , where  $d$  is a positive number such that  $U(-d) > 0$ . Similarly,  $w^0(c+nd) = 0$  for  $n=2,3,\dots$ . Hence,  $w^0(x) = 0$  for  $-\infty < x < \infty$ . In case of  $\underline{E}(u_n) > 0$ ,  $\hat{T}$  is an improper random variable. That is,  $\lim_{n \rightarrow \infty} \Pr\{\hat{r}_n > 0\} > 0$ . For sufficiently large  $n$ ,

$$\Pr\{r_n > 0\} \geq \Pr(A) \cdot \Pr\{r_n > 0 | A\} \geq \Pr(A) \cdot \Pr\{\hat{r}_{n-k} > 0\}.$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} \Pr\{r_n > 0\} \geq \Pr(A) \cdot \lim_{n \rightarrow \infty} \Pr\{\hat{r}_{n-k} > 0\} > 0.$$

Hence  $\sum_{n=0}^{\infty} \Pr(T^0 = n) = 1 - \lim_{n \rightarrow \infty} \Pr\{r_n > 0\} < 1$ ; that is,  $T^0$  is also an improper random variable. Therefore,  $\mathcal{E}$  is a transient recurrent event and  $\lim_{n \rightarrow \infty} \Pr\{w_n^0 = 0\} = 0$ . In a similar manner to the above case,  $w^0(x) = 0$  for  $-\infty < x < \infty$ .

The proof is concluded.

The following theorem results directly from Lemmas 2 and 3.

### Theorem 1.

Suppose that (2. 5) and (3. 3) are satisfied. If and only if  $\underline{E}(u_n) < 0$ ,  $W(x) = \lim_{n \rightarrow \infty} W_n(x)$  exists and is an honest d.f. Moreover,  $W(x)$  is independent of the initial d.f. and is the unique d.f. solution of the following integral equation:

$$\begin{aligned} W(x) = & \int_{-\infty}^{x+0} \left\{ W(x-y) [1 - V(y-x|y)] + \int_{-\infty}^{y-x} W(-z) dV(z|y) \right\} dU(y) \\ & + \int_{x+0}^{\infty} \int_{-\infty}^{+0} W(-z) dV(z|y) dU(y), \text{ for } x \geq 0, \\ = & 0, \text{ for } x < 0. \end{aligned}$$

The above theorem leads to a necessary and sufficient condition under which the fixed-cycle traffic light queue has the stationary distribution. Since by (2. 3),

$$v_n = \inf \left\{ K_n(r), \inf_{0 \leq t \leq g} \{ K_n(r+t) + \ell_n - t \} \right\},$$

Condition (3. 3) will be satisfied except possibly for a very light traffic or an artificially controlled one.

Denote by  $m$  the mean service time and by  $\ell$  the mean lost time. It follows from Wald's theorem (see Takács (1962), page 231) that  $\underline{E}(K_n(T)) = \lambda m T$ .

### Theorem 2.

Suppose that (3. 3) is satisfied. The fixed-cycle traffic light queue has the stationary distribution, if and only if

$$\lambda mT + l < g.$$

If the arrival process  $x_n(t)$  has stationary independent increments, then (3. 3) is satisfied, unless  $\lambda$  or  $m$  vanishes. Hence the above theorem includes Darroch's (1964) result as a simple case.

In the above theorems, it is assumed that (3. 3) is satisfied. In the following, a very light traffic or an artificially controlled one which satisfies  $v_n \leq 0$  a.s. is briefly discussed. In fact, the ideal synchronization of traffic lights may compel the traffic arriving at the intersection to satisfy  $v_n \leq 0$  a.s. In this case, by (2. 6),

$$w_n^0(x) = 0 \text{ (for } x < 0), \quad = 1 \text{ (for } x \geq 0), \text{ for all } n.$$

Therefore, if  $\underline{E}(u_n) < 0$ , then by Lemma 2,

$$W(x) = 0 \text{ (for } x < 0), \quad = 1 \text{ (for } x \geq 0).$$

If  $\underline{E}(|u_n|) < \infty$  and  $\underline{E}(u_n) = 0$ , then again,  $W(x) = 0$  (for  $x < 0$ ),  $= 1$  (for  $x \geq 0$ ), since from Chung and Fuchs (1951),  $T$  is a proper random variable. Finally, consider the case of  $\underline{E}(u_n) > 0$ . If  $\Pr\{w_0 + v_0 > 0\} > 0$ , (2. 5) is satisfied and there exists a finite positive integer  $k$  such that  $\Pr\{u_0 > c/k | w_0 + v_0 > 0\} > 0$ , then in a similar manner to the proof of Lemma 3,  $W(x)$  is a dishonest d.f. On the other hand, if  $w_0 + v_0 \leq 0$  a.s., or  $\Pr\{w_0 + v_0 > 0\} > 0$  and  $\Pr\{u_0 \leq 0 | w_0 + v_0 > 0\} = 1$ , then clearly  $W(x) = 0$  (for  $x < 0$ ),  $= 1$  (for  $x \geq 0$ ).

Hence, for the very light traffic and the artificially controlled one, the fixed-cycle traffic light queue has the stationary distribution  $W(x)=0$  (for  $x < 0$ ),  $=1$  (for  $x \geq 0$ ), if  $\lambda mT + l \leq g$ , or if  $\lambda mT + l > g$  and the initial server occupation time  $w_0$  satisfies  $w_0 + v_0 \leq 0$  a.s., or if  $\lambda mT + l > g$  and  $\Pr\{w_0 + v_0 > 0\} > 0$  and  $\Pr\{u_0 \leq 0 | w_0 + v_0 > 0\} = 1$ .

#### 4. 4. A successive approximation of the limiting distribution

In the preceding section, the integral equation for the limiting d.f. has been obtained. Even in a simple case, however, it may be difficult to solve it. In this section, a successive approximation of the limiting d.f. is presented. Suppose that  $\underline{E}(u_n) < 0$ . Let  $\hat{W}(x)$  be the limiting d.f. of the GI/G/1 queueing process:

$\hat{W}_{n+1} = [\hat{W}_n + u_n]^+$ . Therefore,  $\hat{W}(x)$  satisfies the following integral equation:

$$(4. 1) \quad \hat{W}(x) = \int_{-\infty}^x \hat{W}(x-y) dU(y).$$

Newell (1960), Miller (1963), Darroch (1964), McNeil (1968), Siskind (1970) and others used this simplified model to obtain approximate values of the mean stationary delay per vehicle for fixed-cycle traffic light queues.

#### Theorem 3.

Suppose that  $\underline{E}(u_n) < 0$ . Put  $W_0(x) = \hat{W}(x)$  and

$$(4. 2) \quad W_{n+1}(x) = \int_{-\infty}^{x+0} W_n(x-y) [1 - V(y-x|y)] dU(y) \\ + \int_{-\infty}^{\infty} \int_{-\infty}^{[y-x]^-} W_n(-z) dV(z|y) dU(y).$$

Then  $\hat{W}(x) \leq W_1(x) \leq \dots \leq W_n(x) \leq W_{n+1}(x) \leq \dots$

and  $W_n(x)$  converges to the limiting d.f.  $W(x)$ , as  $n$  tends to infinity.



Proof.

The proof is straightforward. It follows from (4. 1) and (4. 2) that

$$\begin{aligned} W_1(x) - \hat{W}(x) = & \int_{-\infty}^{x+0} \int_{-\infty}^{y-x} \{ \hat{W}(-z) - \hat{W}(x-y) \} dV(z|y) dU(y) \\ & + \int_{x+0}^{\infty} \int_{-\infty}^{+0} \hat{W}(-z) dV(z|y) dU(y) \geq 0. \end{aligned}$$

Consequently, in a similar manner to the proof of lemma 1,

$$W_{n+1}(x) \geq W_n(x), \text{ for } n=1, 2, \dots.$$

The latter half of the theorem is deduced directly from Theorem 1.

This theorem indicates that approximate values by several authors are improved successively by (4. 2).

#### 4. 5. Semi-vehicle-actuated traffic light queue

Consider a traffic along a major road into an intersection controlled by a semi-vehicle-actuated traffic light with detectors on a minor road. It is assumed throughout this section that the traffic on the major road is stochastically independent of that on the minor road. A control algorithm is set up for the traffic light and controls the switching of the light according to events occurring in the traffic on the minor road. Therefore the traffic light turns red or green independently of vehicles arriving on the major road. The lengths of red period and green period are random variables, whose stochastic properties are determined by the control algorithm for the traffic light and the arrival process, if necessary, service times and lost times for vehicles arriving on the minor road. However there may be various types of control algorithms. Since the queue on the major road is concerned primarily with stochastic properties of red periods and green periods, the following condition analogous to (A) is made in order to maintain generality:

(A') The lengths of the  $n$ th red period  $r_n$ 's ( $n=0,1,2,\dots$ ) and those of the  $n$ th green period  $g_n$ 's are mutually independent and identically distributed random variables with finite mean and finite variance,

respectively, and independent each other.

Denote by  $r$  and  $g$  the mean lengths of the red period and the green period. Consequently the lengths of the  $n$ th cycle  $T_n$ 's ( $n=0,1,2,\dots$ ) are mutually independent and identically distributed random variables with finite mean  $T(=r+g)$ .

(B') The stream of vehicles on the major road is a stationary one without after-effects and has finite intensity  $\lambda$ .

Moreover suppose that the same Conditions (C) through (E) as in Section 4. 2 are satisfied. The same notations as in Section 4. 2 are used in the sequel. The following recurrence relation between  $w_n$  and  $w_{n+1}$  ( $n=0,1,2,\dots$ ) is obtained in a similar manner to (2. 1):

$$\begin{aligned}
 (5. 1) \quad w_{n+1} &= w_n + K_n(T_n) - [g_n - l_n]^+, \\
 &\quad \text{if } w_n + \inf\{K_n(r_n), \inf_{0 < t \leq g} \{K_n(r_n + t) + l_n - t\}, \\
 &\quad \quad K_n(T_n) - [g_n - l_n]^+\} > 0, \\
 &= 0 \quad , \quad \text{otherwise.}
 \end{aligned}$$

It is to be noted that the above recurrence relation holds not only for vehicles going straight on or turning left but also for ones turning right. Put

$$(5. 2) \quad u_n = K_n(T_n) - [g_n - l_n]^+$$

$$(5. 3) \quad \text{and } v_n = \inf\{K_n(r_n), \inf_{0 < t \leq g} \{K_n(r_n + t) + l_n - t\},$$

$$K_n(T_n) - [g_n - l_n]^+ \}.$$

Clearly,  $\{u_n\}$  and  $\{v_n\}$  are sequences of mutually independent and identically distributed random variables with finite mean and satisfy (2. 4). Hence the semi-vehicles-actuated traffic light queue is reduced to model (2. 6). However, (2. 5) is not always satisfied for the semi-vehicle-actuated traffic light queue. Moreover, from Wald's theorem

$$\underline{E}(u_n) = \lambda m T - \underline{E}\{[g_n - l_n]^+\}$$

and (3. 3) is satisfied, unless  $\lambda$  or  $m$  vanishes. It can be assumed without loss of generality that  $\lambda$  and  $m$  are positive numbers. Since by the central limit theorem,  $\sum_{i=0}^n T_i \rightarrow \infty$  a.s. if and only if  $n \rightarrow \infty$ , Lemma 2 and Theorem 1 lead immediately to the following theorem.

Theorem 4.

The semi-vehicle-actuated traffic light queue has the stationary distribution, if

$$\lambda m T < \underline{E}\{[g_n - l_n]^+\}$$

and only if, under Condition (2. 5), the above condition is satisfied.

Suppose that

$$(5. 4) \quad g_n \geq l_n \quad \text{a.s.}$$

This assumption appears to be satisfied in practical situations. Then Theorem 4 reduces to:

Corollary 1

The semi-vehicle-actuated traffic light queue has the stationary distribution, if

$$\lambda mT + l < g$$

and only if, under Condition (2. 5), the above condition is satisfied.

To maintain generality of the control algorithm of the traffic light, the arrival process on the minor road and so on, only one Condition (A') has been made for them. In the following, two typical examples of semi-vehicle-actuated traffic light queues are discussed which satisfy Condition (A'). To begin with, suppose that Conditions (C) through (E) are satisfied for vehicles arriving on the minor road. Denote by  $m'$  and  $l'$  the mean service time and the mean lost time, respectively.

Moreover it is assumed that service times and lost times have finite variances and that:

- (F) The traffic light stays red during the discharge time of the queue on the minor road and turns green at the beginning of the first time interval of fixed length  $a$  ( $\geq 0$ ) during which no vehicles arrive on the minor road. It turns red at fixed time  $b$  ( $\geq 0$ ) after any vehicle arrives on the minor road.

(G) The stream of vehicles arriving on the minor road is a compound Poisson process with mean arrival rate  $\lambda(\lambda m' < 1)$  and mean interarrival time  $1/\mu'$ .

Since the traffic stream of the minor road has stationary independent increments with finite variance (see Jánosy, Rényi and Aczél (1950) or Parzen (1962), page 130), Conditions (C) through (G) imply that Condition (A') is satisfied. It follows from Gaver (1959) and Darroch, Newell and Morris (1964) that

$$g = a + b + 1/\mu'$$

and

$$r = m' \{1 + \lambda(b + \ell')\} / (1 - \lambda m') + \{\exp(\mu' a) - (1 + \mu' a)\} / \mu'.$$

Therefore, from Corollary 1, the semi-vehicle-actuated traffic light queue which satisfies (C) to (G) has the stationary distribution,

$$\text{if } \lambda m \mu' m' \{1 + \lambda(b + \ell')\} / (1 - \lambda m') + \lambda m \{\exp(\mu' a) + \mu' b\} < 1 + \mu'(a + b - \ell').$$

As the other example, suppose that the control algorithm is:

(F') The traffic light stays red for the time interval of fixed length  $r$  and turns green. It turns red at fixed time  $b$  ( $\geq 0$ ) after any vehicle arrives on the minor road.

Moreover, in stead of Conditions (C) through (E), suppose

that the queue on the minor road discharges a.s. during one red period. Then Conditions (F') and (G) imply that Condition (A') is satisfied. Clearly,

$$g = b + 1/\mu'.$$

Consequently, from Corollary 1, the above semi-vehicle-actuated traffic light queue has the stationary distribution, if

$$\mu'(\lambda m r + l) < (1 - \lambda m)(1 + \mu' b).$$

Consider the intersection and vehicles arriving on the minor road as the pedestrian crossing controlled by a traffic light with detectors or push-buttons and arriving pedestrians, respectively. Then the above example gives a good approximation to the queue of vehicles in front of the pedestrian crossing.

#### 4. 6. Concluding remarks

In this chapter, the fixed-cycle traffic light queue and the semi-vehicle-actuated one have been reduced to the generalized model (2. 6) of the GI/G/1 queueing process. The vehicle-actuated traffic light queue discussed by Darroch, Newell and Morris (1964) also is reduced to model (2. 6) with  $w_0 + v_0 \leq 0$  a.s. and  $v_n \leq 0$  a.s.

In this chapter,  $\{u_n\}$  and  $\{v_n\}$  are assumed to be sequences of mutually independent and identically distributed random variables. If they may be sequences of dependent random variables, a variety of practical vehicle-actuated traffic light queues may be reduced to model (2. 6) with (2. 4). It is hoped to investigate this model.



## Chapter 5. Traffic Light Queues with Dependent Arrivals

### 5. 1. Introduction

In the preceding chapter, it has been shown that a rather general fixed-cycle traffic light queue reduces to a generalized model of Lindley's (1952) for the GI/G/1 queueing process, and a necessary and sufficient condition is derived under which a stationary distribution exists. These results are based on the condition that the arrival processes in green periods are mutually independent and identically distributed. This condition seems quite practical, especially for arrivals at an isolated signalized intersection. Arrivals at a usual intersection, however, are influenced by traffic lights of other intersections, upstream bottlenecks and so on. Hence it is open to question whether arrivals are independent. In this chapter this condition is dropped and it is assumed that the arrival process is strictly stationary and the number of vehicles arriving in each cycle is ergodic or metrically transitive (see Condition (B) in the next section). The main purpose of this chapter is to show that a fixed-cycle traffic light queue satisfying this assumption on arrivals reduces to a generalized model of Loynes' (1962) and

to derive a sufficient condition under which a stationary distribution exists.

In the preceding chapter, a rather general semi-vehicle-actuated traffic light queue also has been dealt with. In this chapter, under the weakened assumption similar to that of the fixed-cycle traffic light queue, a semi-vehicle-actuated traffic light queue is discussed and a sufficient condition is derived under which a limiting distribution exists. Finally two typical examples of semi-vehicle-actuated traffic light queues are discussed.

## 5. 2. Fixed-cycle traffic light queue

Consider a traffic into an intersection controlled by a fixed-cycle traffic light. Suppose that the following five conditions are satisfied.

- (A) One cycle of the traffic light comprises one (effectively) red period of fixed length  $r$  and one successive (effectively) green period of fixed length  $g$ .

Denote by  $T(=r+g)$  the cycle length and by  $x_n(t)$  ( $0 < t \leq T$ ,  $n=0,1,2,\dots$ ) the number of vehicles which arrive at the intersection during the time interval of length  $t$  from the beginning of the  $n$ th cycle, i.e. the time interval  $(nT, nT+t]$ . Vehicles arriving at the intersection during the  $n$ th red period have to wait till the beginning of the  $n$ th green period at the shortest. Therefore if stochastic properties of  $x_n(r)$  are given, then those of  $x_n(t)$  ( $0 < t < r$ ) are unnecessary for the following analysis. That is, it suffices for the following analysis to specify stochastic properties of the arrival process in the  $n$ th green period  $x_n(r+t)$  ( $0 \leq t \leq g$ ). The sequence of arrival processes in green periods  $\{x_n(t)(r \leq t \leq T); n=0,1,2,\dots\}$  is called strictly stationary, if for arbitrary nonnegative integers  $k_n$  ( $n=0,1,2,\dots$ ) and arbitrarily fixed  $t_{ni}$  ( $n=0,1,2,\dots, i=1,2,\dots,k_n$ ) such that  $r \leq t_{n1} \leq \dots \leq t_{nk_n} \leq T$ , the sequence of random vectors  $\{(x_n(r), x_n(t_{n1}), \dots,$

$x_n(t_{nk_n}), x_n(T)\}$  is strictly stationary. This means that for  $\ell < m < \dots < n$ , any nonnegative integer  $h$  and arbitrary set  $A, B, C$ ,

$$\begin{aligned} & \Pr\{(x_\ell(r), x_\ell(t_{\ell 1}), \dots, x_\ell(t_{\ell k_\ell}), x_\ell(T)) \in A, (x_m(r), \\ & x_m(t_{m1}), \dots, x_m(t_{mk_m}), x_m(T)) \in B, \dots, (x_n(r), \\ & x_n(t_{n1}), \dots, x_n(t_{nk_n}), x_n(T)) \in C\} \\ &= \Pr\{(x_{\ell+h}(r), x_{\ell+h}(t_{\ell 1}), \dots, x_{\ell+h}(t_{\ell k_\ell}), \\ & x_{\ell+h}(T)) \in A, (x_{m+h}(r), x_{m+h}(t_{m1}), \dots, x_{m+h}(t_{mk_m}), \\ & x_m(T)) \in B, \dots, (x_{n+h}(r), x_{n+h}(t_{n1}), \dots, \\ & x_{n+h}(t_{nk_n}), x_{n+h}(T)) \in C\}. \end{aligned}$$

In particular, for  $n=1, 2, \dots$ ,

$$\begin{aligned} & \Pr\{(x_0(r), x_0(t_{01}), \dots, x_0(t_{0k_0}), x_0(T)) \in A\} \\ &= \Pr\{(x_n(r), x_n(t_{01}), \dots, x_n(t_{0k_0}), x_n(T)) \in A\}. \end{aligned}$$

Let  $E(\cdot)$  be the expectation of  $\cdot$ .

- (B) The sequence of arrival processes in green periods  $\{x_n(r+t) (0 \leq t \leq g); n=0, 1, 2, \dots\}$  is strictly stationary. In particular,  $\{x_n(T)\}$  is ergodic or metrically transitive (see Doob (1953), page 457) and  $E\{x_n(T)\} < \infty$ .
- (C) Service times of arriving vehicles and lost times

in all cycles are mutually independent and identically distributed random variables with finite means, independent each other and independent of the arrival processes.

These two conditions seem reasonable and practical.

(D) Once the queue discharges, it almost surely (a.s.) does not reform during the remaining green period.

This condition represents the most distinctive feature of the traffic light queue.

(E) The service discipline is "first-come, first-served" and "preemptive resume".

It is to be noted that except for Condition (B), the same assumptions as in the preceding chapter are made.

Let  $s_{ni}$  and  $\ell_n$  be the service time of the  $i$ th vehicle arriving in the  $n$ th cycle and the lost time in the  $n$ th cycle, respectively. Clearly  $\ell_n \leq g$  a.s. Denote by  $K_n(t)$  the total service time of vehicles arriving in the time interval  $t$  ( $0 < t \leq T$ ) from the beginning of the  $n$ th cycle. Let  $w_n$  be the server occupation time at the beginning of the  $n$ th cycle. Then the same recurrence relation between  $w_n$  and  $w_{n+1}$  ( $n=0,1,2,\dots$ ) as in the preceding chapter holds as follows:

$$(2.1) \quad w_{n+1} = w_n + K_n(T) - (g - \ell_n), \text{ if } w_n + \inf\{K_n(r), \\ \inf_{0 < t \leq g} \{K_n(r+t) + \ell_n - t\}\} > 0, \\ = 0, \text{ otherwise.}$$

It should be noted that this relation holds both for vehicles going straight on or turning left and for ones turning right, although these two types of vehicles are separately dealt with. Put for  $n=0,1,2,\dots$ ,

$$(2.2) \quad u_n = K_n(T) - (g - \ell_n)$$

$$(2.3) \text{ and } v_n = \inf\{K_n(r), \inf_{0 < t \leq g} \{K_n(r+t) + \ell_n - t\}\}.$$

Since without loss of generality  $K_n(r+t)$  can be assumed to be separable (see Doob (1953), page 57),  $v_n$  is well-defined. It is evident that

$$(2.4) \quad u_n \geq v_n \quad \text{a.s.}$$

Moreover the following lemma holds.

Lemma 1.

If Conditions (B) and (C) are satisfied, then the sequence  $\{(u_n, v_n)\}$  defined by (2.2) and (2.3) is strictly stationary,  $\{u_n\}$  is ergodic and  $E\{|u_n|\} < \infty$ .

Proof.

$$\text{Since } K_n(r+t) = \sum_{i=1}^{x_n(r+t)} s_{ni},$$

it follows from Condition (C) that

$$\Pr\{K_n(r+t) \leq x\} = \sum_{j=0}^{\infty} \Pr\left\{\sum_{i=1}^j s_{ni} \leq x\right\} \Pr\{x_n(r+t) = j\}.$$

Consequently Condition (B) compels that for arbitrarily fixed sequence  $\{t_{ni}\}$  such that  $0 < t_{n1} < \dots < t_{ni} < \dots < g$ , the sequence of random vectors  $\{(K_n(r), K_n(r+t_{n1}), \dots, K_n(r+t_{ni}))$ ,

$\dots, K_n(T))$  is strictly stationary. Furthermore, the separability of  $K_n(r+t)$  implies that  $v_n = \inf\{K_n(r), \inf_{\{t_{ni}\}}\{K_n(r+t_{ni}) + \ell_n - t_{ni}\}\}$  a.s., where  $\{t_{ni}\}$  is a sequence satisfying the conditions of the separability condition. Since  $\{\ell_n\}$  is a sequence of mutually independent and identically distributed random variables and independent of  $\{K_n(r+t)\}$ ,  $\{(u_n, v_n)\}$  is strictly stationary by (2. 2) and (2. 3). In order to prove the ergodicity of  $\{u_n\}$  it suffices to show that  $\{K_n(T)\}$  is ergodic, because  $\{\ell_n\}$  is ergodic and independent of  $\{K_n(T)\}$  (see Doob (1953), page 460). The proof is based on the necessary and sufficient condition for ergodicity (see Billingsley (1965), page 17). Let A and B be arbitrary Borel measurable sets of i dimensional Euclidian space and j dimensional Euclidian space. For positive integers  $k_1 < k_2 < \dots < k_i$  and  $\ell_1 < \ell_2 < \dots < \ell_j$ ,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \Pr\{(K_{k_1}(T), \dots, K_{k_i}(T)) \in A, (K_{h+\ell_1}(T), \\
 & \quad \dots, K_{h+\ell_j}(T)) \in B\} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \sum_{\{m_1, \dots, m_i, n_1, \dots, n_j\}} \Pr\left\{\left(\sum_{i=1}^{m_1} s_{k_1 i}, \dots, \sum_{i=1}^{m_i} s_{k_i i}\right) \in A, \left(\sum_{i=1}^{n_1} s_{h+\ell_1 i}, \dots, \sum_{i=1}^{n_j} s_{h+\ell_j i}\right) \in B\right\} \\
 & \Pr\{x_{k_1}(T) = m_1, \dots, x_{k_i}(T) = m_i, x_{h+\ell_1}(T) = n_1, \dots,
 \end{aligned}$$

$$\begin{aligned}
& x_{h+\ell_j}(T)=n_j \} \\
& = \sum_{\{m_1, \dots, m_i, n_1, \dots, n_j\}} \Pr\left\{\left(\sum_{i=1}^{m_1} s_{k_1 i}, \dots, \right.\right. \\
& \quad \left.\left.\sum_{i=1}^{m_i} s_{k_i i}\right) \in A\right\} \Pr\left\{\left(\sum_{i=1}^{n_1} s_{\ell_1 i}, \dots, \sum_{i=1}^{n_j} s_{\ell_j i}\right) \in B\right\} \\
& \quad \Pr\{x_{k_1}(T)=m_1, \dots, x_{k_i}(T)=m_i\} \Pr\{x_{\ell_1}(T)=n_1, \\
& \quad \dots, x_{\ell_j}(T)=n_j\} \\
& = \Pr\{(K_{k_1}(T), \dots, K_{k_i}(T)) \in A\} \Pr\{(K_{\ell_1}(T), \dots, \\
& \quad K_{\ell_j}(T)) \in B\}.
\end{aligned}$$

This means  $\{K_n(T)\}$  is ergodic. It is clear by Conditions (B) and (C) that  $E\{|u_n|\} < \infty$ . The proof is concluded.

Therefore the fixed-cycle traffic light queue which satisfies Conditions (A) through (E) is reduced to the following general model: for  $n=0,1,2,\dots$ ,

$$\begin{aligned}
(2.5) \quad w_{n+1} &= w_n + u_n, \text{ if } w_n + v_n > 0, \\
&= 0, \text{ otherwise,}
\end{aligned}$$

where  $\{(u_n, v_n)\}$  is strictly stationary and  $\{u_n\}$  is ergodic. This recurrence relation was discussed in the preceding chapter, where  $\{(u_n, v_n)\}$  was a sequence of mutually independent and identically distributed random variables. The previous model is an extension of



Lindley's (1952). While Loynes (1962) investigated a different extension to Lindley's model, expressed by  $w_{n+1}=[w_n+u_n]^+$ , where  $[x]^+=\max(0, x)$  and  $\{u_n\}$  was strictly stationary and ergodic. Hence the above model (2. 5) is a generalization to Loynes' model.

Let  $W_n(x)=\Pr\{w_n \leq x\}$ . Denote by  $w_n^0$  the random variable  $w_n$  conditioned by  $w_0=0$  a.s. and by  $W_n^0(x)$  its distribution function. In the following the above model (2. 5) with (2. 4) is dealt with. The following lemma is prepared in order to derive a sufficient condition for the stationary distribution of  $\{w_n\}$  to exist.

Lemma 2.

As  $n$  tends to infinity,  $W_n^0(x)$  converges monotonically to an honest or dishonest distribution function  $W^0(x)$ .

Proof.

This lemma can be proved in much the same way as Lemma 1 of Loynes (1962). According to Doob (1953, page 456), the strictly stationary one-sided random sequence  $\{(u_n, v_n); n=0,1,2,\dots\}$  can be extended to the strictly stationary two-sided random sequence  $\{(u_n, v_n); n=\dots,-1,0,1,\dots\}$  without any change of the joint distribution of the random variables  $(u_0, v_0), \dots, (u_n, v_n)$ . For this two-sided random sequence  $\{(u_n, v_n)\}$  and nonnegative integer  $m$ , put for  $n \leq -m$

$$w_n^{(m)}=0 \quad \text{a.s.}$$

and for  $n \geq -m$

$$w_{n+1}^{(m)} = w_n^{(m)} + u_n, \text{ if } w_n^{(m)} + v_n > 0, \\ = 0, \text{ otherwise.}$$

Let  $I\{\cdot\}$  be the characteristic function of event  $\cdot$ .

Since

$$w_n^{(m+1)} = w_n^{(m)} = 0 \text{ for } n < -m$$

and by (2.4)  $w_n^{(m+1)} = I\{v_n > 0\} u_n \geq 0 = w_n^{(m)}$  for  $n = -m$ , it follows that

$$w_n^{(m+1)} \geq w_n^{(m)} \text{ a.s. for } n \leq -m.$$

If  $w_n^{(m+1)} \geq w_n^{(m)}$  a.s. for some  $n$ ,

$$\begin{aligned} \text{then } w_{n+1}^{(m+1)} - w_{n+1}^{(m)} \\ = I\{w_n^{(m+1)} + v_n > 0\} (w_n^{(m+1)} + u_n) - I\{w_n^{(m)} + v_n > 0\} (w_n^{(m)} + u_n) \\ = I\{w_n^{(m+1)} + v_n > 0\} (w_n^{(m+1)} - w_n^{(m)}) \\ + I\{w_n^{(m+1)} + v_n > 0, w_n^{(m)} + v_n \leq 0\} (w_n^{(m)} + u_n) \geq 0 \text{ a.s.} \end{aligned}$$

Consequently by induction,  $w_n^{(m+1)} \geq w_n^{(m)}$  a.s. for all  $m$  and  $n$ . Therefore  $\{w_0^{(n)}; n=0,1,2,\dots\}$  is a monotone nondecreasing random sequence and as  $n$  tends to infinity,  $w_0^{(n)}$  converges to a proper or improper random variable. While from the strict stationarity of  $\{(u_n, v_n)\}$ , the distribution function  $w_n^0(x)$  of the random variable  $w_n^0 = w_n^{(0)}$  is the same as that of  $w_0^{(n)}$ . Hence  $w_n^0(x)$  converges monotonically to an honest or dishonest distribution function  $w^0(x)$ , as  $n$

tends to infinity.

Theorem 1.

If  $\underline{E}(u_n) < 0$ , then  $W_n(x)$  converges to the unique honest distribution function independently of the initial distribution, as  $n$  tends to infinity.

Proof.

To begin with, it will be proved that  $W^0(x)$  is the honest distribution function. Let  $S_{ni} = \sum_{j=1}^n u_j$ , for  $i=0,1,2,\dots,n$ . Clearly by (2.5),  $w_1^0 \leq [u_0]^+ \text{ a.s.}$  If  $w_n^0 \leq [\sup_i S_{n-1i}]^+ \text{ a.s.}$ , then by (2.4),

$$\begin{aligned} w_{n+1}^0 &= I\{w_n^0 + v_n > 0\}(w_n^0 + u_n) \leq I\{w_n^0 + u_n > 0\}(w_n^0 + u_n) \\ &\leq [[\sup_i S_{n-1i}]^+ + u_n]^+ = [\sup_i S_{ni}]^+ \text{ a.s.} \end{aligned}$$

Therefore by induction, for all  $n=0,1,2,\dots$ ,

$$w_n^0 \leq [\sup_i S_{n-1i}]^+ \text{ a.s.}$$

Since from the ergodic theorem (see, for example, Billingsley (1965), page 13),

$$\lim_{n \rightarrow \infty} S_{ni}/(n-i+1) = \underline{E}(u_0) < 0 \text{ a.s.}$$

for arbitrarily fixed  $i$ ,  $[\sup_i S_{n-1i}]^+$  is a.s. finite for all  $n$ . Consequently from the monotone convergence of  $W_n^0(x)$ ,  $W^0(x)$  is the unique honest distribution function. Define the first passage time  $T$  of  $w_n$  starting from any proper random variable  $w_0$  to zero as

$T=0$  if  $w_0=0$ , otherwise

$$T=\{n; w_0 + \min_{i \leq n-2} [S_{i-10} + v_i] > 0, w_0 + S_{n-20} + v_{n-1} \leq 0\},$$

where for  $i < 0$ ,  $S_{i0} \equiv 0$  and  $v_i \equiv 0$ . It is obvious that

$$\sum_{n=0}^{\infty} \Pr\{T=n\} = 1 - \lim_{n \rightarrow \infty} \Pr\{w_0 + \inf_{i \leq n} [S_{i-10} + v_i] > 0\}.$$

Since  $S_{n0} \rightarrow -\infty$  a.s. as  $n \rightarrow \infty$ ,  $T$  is a proper random variable.

Suppose that  $w_n \geq w_n^0$  a.s. Then

$$\begin{aligned} w_{n+1} - w_{n+1}^0 &= I\{w_n^0 + v_n > 0\}(w_n - w_n^0) + I\{w_n + v_n > 0, \\ &\quad w_n^0 + v_n \leq 0\}(w_n + v_n) \\ &\geq 0 \text{ a.s.} \end{aligned}$$

By induction, for all  $n=0,1,2,\dots$ ,  $w_n \geq w_n^0$  a.s., because  $w_0 \geq 0 \equiv w_0^0$  a.s. Therefore on the event  $\{T=n\}$ ,  $w_n = w_n^0 = 0$  and then  $w_k = w_k^0$  for all  $k=n+1,\dots$ . Since  $T$  is a proper random variable,  $W_n(x)$  converges to  $W^0(x)$  independently of the initial distribution, as  $n$  tends to infinity.

The proof is concluded.

It should be remarked that this theorem holds on the condition that, instead of the ergodicity of  $\{u_n\}$ ,  $\{u_n\}$  obeys the strong law of large numbers, since in the above proof, the former is used to assure the latter. Denote by  $\lambda T$  the mean number of vehicles arriving in one cycle, by  $m$  the mean service time and by  $\ell$  the mean lost time. It follows from Condition (C) and Wald's theorem

(see Takács (1962), page 231) that  $E(K_n(T)) = \lambda mT$ . The above theorem leads directly to a sufficient condition under which the fixed-cycle traffic light queue has the stationary distribution.

Theorem 2.

If  $\lambda mT + l < g$ , then the fixed-cycle traffic light queue which satisfies Conditions (A) through (E) has a unique stationary distribution independent of any initial distribution.

### 5. 3. Semi-vehicle-actuated traffic light queue

Consider traffic along a major road into an intersection controlled by a semi-vehicle-actuated traffic light with detectors on a minor road. Let  $r_n$ ,  $g_n$  and  $T_n$  ( $n=0,1,2,\dots$ ) be the lengths of the  $n$ th red period, the  $n$ th green period and the  $n$ th cycle, respectively. In order to maintain generality, the following assumptions are made which are weaker than the corresponding ones in the preceding chapter:

- (A') The sequence of positive random vectors  $\{(T_n, g_n); n=0,1,2,\dots\}$  is strictly stationary and ergodic, and  $\underline{E}(T_n)=T<\infty$ .

Let  $r$  be the mean length of the red period and  $g$  that of the green period. Suppose that the 0th cycle begins at time zero. Denote by  $x(t)$  ( $t>0$ ) and  $x_m$  the numbers of vehicles on the major road in time interval  $(0, t]$  and in the  $m$ th unit time interval  $(m, m+1]$ . It is proper to assume that the traffic light turns red or green independently of vehicles on the major road.

- (B') The arrival process  $x(t)$  on the major road is stationary and independent of  $\{(T_n, g_n)\}$ . In particular the sequence  $\{x_m\}$  is ergodic with  $\underline{E}(x_m)=\lambda (\neq 0) < \infty$ .

The stationarity of the arrival process means that the joint distribution of arrivals in a set of arbitrarily

fixed time intervals is invariant under translation (Khinchine (1960), page 24).

(C') Service times of vehicles arriving on the major road and lost times in all cycles are mutually independent and identically distributed random variables with finite means, independent each other and independent of  $\{(T_n, g_n)\}$  and the arrival process.

Moreover suppose that the same Conditions (D) and (E) as in the preceding section are satisfied. The notation similar to that in the preceding section is used in the sequel. Since if  $g_n \leq l_n$ , then  $w_{n+1} = w_n + K_n(T_n)$ , it follows from Conditions (A'), (D) and (E) that the following recurrence relation between  $w_n$  and  $w_{n+1}$  ( $n=0, 1, 2, \dots$ ) is obtained in a similar way to (2. 1):

$$(3. 1) \quad w_{n+1} = w_n + K_n(T_n) - [g_n - l_n]^+, \text{ if } w_n + \inf\{K_n(r_n), \\ \inf_{0 < t < g_n} \{K_n(r_n + t) + l_n - t\}, K_n(T_n) - [g_n - l_n]^+\} > 0, \\ = 0, \text{ otherwise.}$$

Put

$$(3. 2) \quad u_n = K_n(T_n) - [g_n - l_n]^+$$

$$(3. 3) \text{ and } v_n = \inf\{K_n(r_n), \inf_{0 < t < g_n} \{K_n(r_n + t) + l_n - t\}, \\ K_n(T_n) - [g_n - l_n]^+\}.$$

Clearly  $v_n$  is well-defined and  $u_n$  and  $v_n$  satisfy Relation (2. 4). Furthermore the following lemma holds:

Lemma 3.

If Conditions (A'), (B') and (C') are satisfied, then the sequence  $\{(u_n, v_n)\}$  defined by (3. 2) and (3. 3) is strictly stationary and  $\{u_n\}$  obeys the strong law of large numbers.

Proof.

It follows from Condition (A') that  $\{(r_n, g_n)\}$  and  $\{(T_n, r_n)\}$  are strictly stationary. Moreover from Condition (B') it follows that for any positive integer  $k$ , any nonnegative numbers  $t, u, t_1 < t_2 < \dots < t_k$  and arbitrary  $k$ -dimensional closed set  $A$ ,

$$\begin{aligned} & \Pr\{(x(t+t_1)-x(t), x(t+t_2)-x(t), \dots, x(t+t_k)-x(t)) \in A\} \\ &= \Pr\{(x(u+t_1)-x(u), x(u+t_2)-x(u), \dots, x(u+t_k)-x(u)) \in A\}. \end{aligned}$$

Besides for  $0 \leq t \leq g_n$ ,

$$x_n(r_n+t) = x\left(\sum_{i=0}^{n-1} T_i + r_n + t\right) - x\left(\sum_{i=0}^{n-1} T_i\right) \text{ a.s.}$$

Therefore for  $\ell < m < \dots < n$  and any integer  $h$ , the joint distribution of  $x_\ell(r_\ell+t)(0 \leq t \leq g_\ell)$ ,  $x_m(r_m+t)(0 \leq t \leq g_m)$ , ... and  $x_n(r_n+t)(0 \leq t \leq g_n)$  conditioned by values of

$\sum_{i=0}^{\ell-1} T_i, r_\ell, g_\ell, \sum_{i=\ell+1}^{m-1} T_i, r_m, g_m, \dots, r_n$  and  $g_n$  is equal to the joint distribution of  $x_{\ell+h}(r_{\ell+h}+t)(0 \leq t \leq g_{\ell+h})$ ,  $x_{m+h}(r_{m+h}+t)(0 \leq t \leq g_{m+h})$ , ... and  $x_{n+h}(r_{n+h}+t)(0 \leq t \leq g_{n+h})$



for any  $\sum_{i=0}^{h-1} T_i$  conditioned by the same values of  $\sum_{i=0}^{l-1} T_{i+h}$ ,  $r_{l+h}$ ,  $g_{l+h}$ ,  $\sum_{i=l+1}^{m-1} T_{i+h}$ ,  $r_{m+h}$ ,  $g_{m+h}$ , ...,  $r_{n+h}$  and  $g_{n+h}$ . On account of Condition (A'), the sequence  $\{x_n(r_n+t) (0 \leq t \leq g_n)\}$  is strictly stationary. Hence in a similar way to the proof of Lemma 1,  $\{(u_n, v_n)\}$  is strictly stationary. From Conditions (A') and (C'), it suffices to prove that  $\{K_n(T_n)\}$  obeys the strong law of large numbers, in order to prove  $u_n$  obeys this law. Let  $q_n = [\sum_{i=0}^{n-1} T_i]$ ,  $y_n = \sum_{i=0}^{q_n} x_i$  and  $y'_n = \sum_{i=0}^{n-1} x_i(T_i)$ , where  $[\cdot]$  means the maximal integer not exceeding  $\cdot$ . By Condition (A'),  $\lim_{n \rightarrow \infty} q_n/n = \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \sum_{i=0}^{n-1} T_i + (q_n - \sum_{i=0}^{n-1} T_i) \right\} = \underline{E}(T_n) = T$  a.s. Therefore by Conditions (B') and (C'),  $\lim_{n \rightarrow \infty} y_n/q_n = \underline{E}(x_i) = \lambda$  a.s. and  $\lim_{n \rightarrow \infty} \sum_{j=1}^n s.j / y_n = \underline{E}(s.j) = m$  a.s. Consequently,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} K_i(T_i) &= \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \sum_{j=1}^{y_n} s.j + \sum_{j=y_n+1}^{y'_n} s.j \right\} \\ &= \lim_{n \rightarrow \infty} \frac{q_n}{n} \cdot \frac{y_n}{q_n} \cdot \frac{1}{y_n} \sum_{j=1}^{y_n} s.j \\ &= \lambda m T = \underline{E}\{K_n(T_n)\}. \end{aligned}$$

The proof is concluded.

Hence the semi-vehicle-actuated traffic light queue is reduced to (2. 5) in which  $\{(u_n, v_n)\}$  is strictly stationary and satisfies (2. 4), and  $\{u_n\}$  obeys the strong law of large numbers. It is clear that  $\underline{E}(u_n) = \lambda m T - \underline{E}\{[g_n - l_n]^+\}$ . The following theorem results

directly from Theorem 1 and the remark following it.

Theorem 3.

If  $\lambda mT < E\{[g_n - l_n]^+\}$ , then the semi-vehicle-actuated traffic light queue which satisfies Conditions (A') through (C') and (D), (E) has a unique limiting distribution independent of any initial distribution.

Under the assumption  $g_n \geq l_n$  a.s., it has been shown in the preceding chapter that the condition  $\lambda mT + l < g$  is sufficient for the stationary distribution to exist. If this condition is satisfied, then  $\lambda mT < E\{[g_n - l_n]^+\}$ , because  $[g_n - l_n]^+ \geq g_n - l_n$  a.s.

Corollary 1.

If  $\lambda mT + l < g$ , then the semi-vehicle-actuated traffic light queue which satisfies Conditions (A') through (C') and (D), (E) has a unique limiting distribution independent of any initial distribution.

To maintain generality of the control algorithm of the traffic light, the arrival process on the minor road and so on, only one Condition (A') has been made for them. In the sequel, two typical examples of semi-vehicle-actuated traffic light queues are discussed which satisfy Condition (A'). The first example satisfies Condition (C') for vehicles arriving on the minor road and the following assumptions:

(F) The traffic light stays red during the discharge

time of the queue on the minor road and turns green if no vehicles arrive at the end of that discharge time. It turns red at the instant any vehicle arrives on the minor road.

- (G) The interarrival times of vehicles on the minor road are mutually independent and identically distributed random variables with finite positive mean  $\lambda$  such that  $m'/\lambda < 1$  for the mean service time  $m'$ .

Note that the green period of the traffic light corresponds to the idle period of the GI/G/1 queue on the minor road, that is, the positive time interval during which the GI/G/1 queue on the minor is empty: on the other hand, the red period corresponds to the residual busy period, that is, the time interval between the end of the idle period and the first instant thereafter at which the queue becomes empty under the condition that the first vehicle has to wait till the end of the lost time.

Therefore  $\{(T_n, g_n)\}$  is a sequence of mutually independent and identically distributed random vectors; this fact implies Condition (A') is satisfied. Denote by  $\sigma_n$  and  $\tau_n$  ( $n=1, 2, \dots$ ) the interarrival time between the  $n$ th and the  $(n+1)$ th vehicles arriving on the minor road and the service time of the  $n$ th vehicle, respectively. Let  $X_n = \tau_n - \sigma_n$ ,  $S_n = \sum_{i=1}^n X_i$ ,  $y_n = \Pr\{S_n \geq 0\}$  and  $Z_n(x) = \Pr\{S_1 \leq 0, \dots, S_n \leq x\}$ .

$S_2 \leq 0, \dots, S_{n-1} \leq 0, x \leq S_n \leq 0\}$ . Denote by  $\ell'$  and  $L'(\cdot)$  the expectation and the distribution function of the lost time for vehicles on the minor road. Then the mean number  $\underline{E}(N)$  of vehicles passing through the intersection along the minor road during the red period (see Cohen (1969), page 287, or Prabhu (1965), page 147) is:

$$\underline{E}(N) = \exp\left\{\sum_{n=1}^{\infty} y_n/n\right\} \left\{1 + \sum_{n=1}^{\infty} \int_0^{\infty} Z_n(-x) dL'(x)\right\}.$$

Consequently,  $\underline{E}(T_n) = \alpha \exp\left\{\sum_{n=1}^{\infty} y_n/n\right\} \left\{1 + \sum_{n=1}^{\infty} \int_0^{\infty} Z_n(-x) dL'(x)\right\}$

and  $\underline{E}(g_n) = (\alpha - m') \exp\left\{\sum_{n=1}^{\infty} y_n/n\right\} \left\{1 + \sum_{n=1}^{\infty} \int_0^{\infty} Z_n(-x) dL'(x)\right\} - \ell'.$

It follows from Corollary 1 that the semi-vehicle-actuated traffic light queue satisfying Conditions (C'), (F) and (G) has a limiting distribution independent of any initial distribution, if

$$\ell + \ell' < \{(1 - \lambda m) \alpha - m'\} \exp\left\{\sum_{n=1}^{\infty} y_n/n\right\} \left\{1 + \sum_{n=1}^{\infty} \int_0^{\infty} Z_n(-x) dL'(x)\right\}.$$

As the other example, consider a queue of vehicles in front of a pedestrian crossing controlled by a traffic light with detectors or push-buttons for arriving pedestrians. The control algorithm is:

(F') The traffic light stays red for the time interval of fixed length  $r$  and turns green. It turns red at fixed time  $a$  ( $\geq 0$ ) after any pedestrian arrives at the pedestrian crossing.

Moreover the following assumption is made:

(G') Groups of pedestrians arrive at the pedestrian crossing. The interarrival times of successive groups are mutually independent and identically distributed random variables with finite positive mean  $\lambda$ . The queue of pedestrians discharges a.s. during one red period.

Suppose that the first group of pedestrians arrives at time  $(-a)$ . Then each green period comprises one residual life time in renewal theory (Smith (1958)) and one time interval of fixed length  $a$ . Therefore Condition (A') is satisfied and  $E(g_n) = \lambda\{1+H(a+r)\} - r$ , where  $H(\cdot)$  denotes the renewal function for interarrival times of successive groups of pedestrians. Hence it follows from Corollary 1 that the queue in front of the pedestrian crossing satisfying Conditions (F') and (G') has a limiting distribution independent of any initial distribution, if

$$\lambda + r < \lambda(1 - \lambda m)\{1 + H(a+r)\}.$$

#### 5. 4. Concluding remarks

In this chapter sufficient conditions are derived under which the fixed-cycle and the semi-vehicle-actuated traffic light queues have stationary distributions independent of initial distributions. If  $\{v_n\}$  defined by (2. 3) or (3. 3) obeys the strong law of large numbers, then in a similar way to the proof of Theorem 1, it can be proved that  $E(v_n) > 0$  is sufficient for no honest stationary distributions to exist. It may be difficult, however, to find those conditions on the arrival process, service times and lost times under which  $\{v_n\}$  obeys the strong law of large numbers.

## Chapter 6. Traffic Light Queues with Departure Headways Depending Upon Positions

### 6. 1. Introduction

In the preceding two chapters, departure headways of all vehicles except the first have been assumed mutually independent and identically distributed. Anker, Gaffarian and Gray (1968), however, analyzed data on departure headways of vehicles queueing up in a straight-through lane leading into a signalized intersection and showed that departure headways were mutually independent random variables with shifted Erlang density functions depending upon positions. In this chapter, therefore, departure headways are assumed to be mutually independent random variables with general distribution functions dependent upon positions (see Condition (C) in the next section ). It should be noted that under this assumption, the introduction of the starting delay or the lost time loses its effect of simplifying the analysis. In the next section, fixed-cycle traffic light queues are dealt with. For independent arrivals, a necessary and sufficient condition is derived under which a stationary queue length distribution exists, and for dependent arrivals,

a sufficient condition for a stationary queue length distribution to exist is derived. In Section 5. 3, a fixed-cycle traffic light queue with constant departure headways depending upon positions and with stationary and independent arrivals is dealt with and its stationary queue length distribution is obtained. In Section 6. 4, semi-vehicle-actuated traffic light queues are dealt with and sufficient conditions for stationary queue length distributions to exist are derived. As an example, a queue of vehicles in front of a pedestrian crossing is discussed which is controlled by traffic lights with detectors or push-buttons for arriving pedestrians.



## 6. 2. Fixed-cycle traffic light queue

Consider a queue of vehicles in a straight-through lane of an approach to an intersection controlled by a fixed-cycle traffic light. The signal sequence of traffic lights is amber, red, and green in Japan, U.S.A. and some countries: while in U.K. it is amber, red, red and amber shown together, and green. Since the signal sequence varies thus with countries, in order to keep up generality, the time interval from the end of a green period to the beginning of the next green period is called the red period throughout this chapter. To begin with, suppose that the following two natural conditions are satisfied:

- (A) One cycle of the traffic light controlling the queue is composed of one red period of fixed length  $r$  and one successive green period of fixed length  $g$ .

Hence the cycle length of the traffic light is  $T(=r+g)$ . It is to be noted that the green period is not the "effective green period" applied popularly but the exact time interval during which the traffic light shows the signal "green".

- (B) Vehicles which arrive in the green period and find the queue empty pass through the intersection

without delay.

This condition means that once the queue becomes empty in the green period, it remains empty till the end of the green period. This fact is regarded as the most distinctive feature of the traffic light queue.

Let some vehicles wait in the queue at the end of the red period. As the traffic light turns green, the  $i$ th ( $i=1,2,\dots$ ) vehicle in the queue crosses the stop line at time  $t_i$  ( $0 < t_i \leq g$ ) from the beginning of the green period and passes through the intersection. The time interval  $t_i$  is called the departure headway of position 1 or that of the first vehicle and the time interval between  $t_{i-1}$  and  $t_i$  ( $i=2,3,\dots$ ) is called the departure headway of position  $i$  or that of the  $i$ th vehicle. It should be noted that departure headways are defined only for vehicles that stop by the signal "red" or slow down owing to the existing queue. Anker, Gafarian and Gray (1968) collected data on departure headways of vehicles queueing up on a straight-through middle lane of a three-lane road into an intersection with no downstream bottleneck and analyzed carefully statistical properties of departure headways. Their results are that departure headways are mutually independent random variables with shifted Erlang density functions depending upon positions and that departure headways of the positions

from 2 on are almost surely (a.s.) greater than a positive number (see, Table 10 in Anker, Gaffarian and Gray (1968)). Although many authors have investigated traffic light queues, all of them assume that departure headways of the positions from 2 on are constant or, more generally, mutually independent and identically distributed random variables (m.i.i.d.r.v.s). Moreover they adopt the convention that the introduction of the terms "effectively red", "effectively green" and "lost time" makes departure headways of all positions constant or m.i.i.d.r.v.s. The above results based upon experiment, however, show that this conventional assumption is not satisfied in the actual situation. This compels that the following condition should be made on departure headways:

- (C) Departure headways are mutually independent random variables with distribution functions (d.f.s)  $F_i$  depending upon positions  $i$ , and ones from some position on are a.s. greater than a positive number.

This condition is more general than both the conventional assumption and the above-mentioned results obtained empirically, and seems to be in the nature of departure headways. Since the terms "effectively red", "effectively green" and "lost time" are not employed in this chapter,

as remarked in Condition (A), it is essential to make some assumption on the effect of the warning period in the red period: note that "not-green" is called red in this chapter. Suppose that:

- (D) If the queue is not empty at the end of the green period, the vehicle at the head of the queue passes through the intersection with probability  $p$  ( $0 \leq p \leq 1$ ) and stops with probability  $1-p$ .

This assumption may be a first approximation of what actually happens.

Let the  $n$ th cycle of the traffic light begin at time  $nT$  and finish at time  $(n+1)T$ , where  $n=0,1,2,\dots$ . Denote by  $s_{ni}$  the departure headway of position  $i$  ( $i=1,2,\dots$ ) in the  $n$ th cycle. Condition (C) implies that the random variables  $s_{ni}$  ( $n=0,1,2,\dots$ ,  $i=1,2,\dots$ ) are mutually independent and, for fixed  $i$ , the random variables  $s_{ni}$  ( $n=0,1,2,\dots$ ) are identically distributed with d.f.  $F_i$ . Put  $s_{n0}=0$  a.s. for  $n=0,1,2,\dots$  and define the saturation flow  $f_n$  of the  $n$ th green period as

$$(2.1) \quad f_n = \sup \left\{ k; \sum_{i=0}^k s_{ni} \leq g \right\}.$$

Condition (C) implies that the saturation flows  $f_n$  are m.i.i.d.r.v.s and for a sufficiently large finite number  $K$ ,

$$(2.2) \quad f_n < K \quad \text{a.s.}$$

Define the random time  $t_{nm}$  ( $m=0,1,2,\dots$ ) as

$$(2.3) \quad t_{nm} = \inf \left\{ r + \sum_{i=0}^{f_n} s_{ni}, T \right\}.$$

Obviously,  $t_{n0}=r$  and  $t_{nm}=T$  for  $m=f_n+1, f_n+2, \dots$ . For  $m=1,2,\dots, f_n$ , the random variable  $t_{nm}$  represents the time at which the  $m$ th vehicle crosses the stop line during the  $n$ th green period, if this vehicle is in the queue. Moreover, Condition (C) implies that for fixed  $m$ , the random times  $t_{nm}$  ( $n=0,1,2,\dots$ ) are mutually independent and identically distributed (m.i.i.d.) and that for a nonnegative integer  $k$ , the  $k$ -dimensional random vectors  $(t_{n1}, t_{n2}, \dots, t_{nk})$  conditioned by the events  $\{f_n=k\}$  are also m.i.i.d. Let  $x_n(t)$  denote the number of vehicles in the straight-through lane arriving at the intersection in the time interval of length  $t$  from the beginning of the  $n$ th cycle i.e. the time interval  $(nT, nT+t]$ . Define  $y_{nm}$  ( $m=0,1,2,\dots$ ) as follows:

$$(2.4) \quad y_{n0} = x_n(t_{n0}) = x_n(r)$$

and for  $m=1,2,\dots$ ,

$$(2.5) \quad y_{nm} = x_n(t_{nm}) - x_n(t_{nm-1}).$$

These random variables  $y_{n0}, y_{nm}$  ( $m=1,2,\dots, f_n$ ) and

$y_{nf_n+1}$  represent the numbers of vehicles arriving at the intersection in the  $n$ th red period, in the departure headway of position  $m$  and in the remaining green period  $(t_{nf_n}, T]$ , respectively.

Denote by  $L_n$  ( $n=0,1,2,\dots$ ) the queue length at the beginning of the  $n$ th cycle and by  $N_{nm}$  ( $m=0,1,2,\dots$ ) the queue length immediately after the random time  $t_{nm}$ . Then from Conditions (A) and (B) it follows that

$$(2.6) \quad N_{n0} = L_n + y_{n0}$$

and for  $m=1,2,\dots,f_n$ ,

$$(2.7) \quad \begin{aligned} N_{nm} &= N_{nm-1} + y_{nm} - 1, \text{ if } N_{nm-1} > 0, \\ &= 0, \text{ otherwise.} \end{aligned}$$

Let  $\{e_n; n=0,1,2,\dots\}$  be a sequence of mutually independent random variables which take value 0 with probability  $(1-p)$  and value 1 with probability  $p$ . That is,  $\{e_n\}$  is the result of a sequence of Bernoulli trials in Condition (D). Therefore by Conditions (A) and (B) and (2.2),

$$(2.8) \quad \begin{aligned} L_{n+1} &= N_{nf_n+1} = N_{nf_n} + y_{nf_n+1} - e_n, \text{ if } N_{nf_n} > 0, \\ &= 0, \text{ otherwise.} \end{aligned}$$

Combination of Equations (2.6), (2.7) and (2.8) leads to the following expression:

$$(2.9) \quad L_{n+1} = L_n + \sum_{m=0}^{f_n+1} y_{nm} - e_n - f_n,$$

$$\begin{aligned} & \text{if } \min\{N_{n0}, N_{n1}, \dots, N_{nf_n+1}\} > 0, \\ & = 0 \quad , \text{ otherwise.} \end{aligned}$$

Therefore on account of (2. 4) and (2. 5), the recurrence relation between  $L_n$  and  $L_{n+1}$  ( $n=0,1,2,\dots$ ) holds as follows:

$$\begin{aligned} (2. 10) \quad L_{n+1} &= L_n + x_n(T) - e_n - f_n, \\ & \text{if } L_n + \min\left\{\min_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T) - e_n - f_n\right\} > 0, \\ & = 0 \quad , \text{ otherwise.} \end{aligned}$$

Now define random variables  $u_n$  and  $v_n$  ( $n=0,1,2,\dots$ ) as

$$(2. 11) \quad u_n = x_n(T) - e_n - f_n$$

and

$$(2. 12) \quad v_n = \min\left\{\min_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T) - e_n - f_n\right\}.$$

It is clear by these definitions and (2. 2) that

$$(2. 13) \quad u_n \geq v_n \quad \text{a.s.}$$

and

$$(2. 14) \quad v_n \geq -K \quad \text{a.s.}$$

Hence Relation (2. 10) reduces to the following general model: for  $n=0,1,2,\dots$ ,

$$\begin{aligned} (2. 15) \quad L_{n+1} &= L_n + u_n, \text{ if } L_n + v_n > 0, \\ & = 0 \quad , \text{ otherwise,} \end{aligned}$$

where  $u_n$  and  $v_n$  satisfy (2. 13) and (2. 14). It should be noted that the redundant term  $N_{nf_n} + 1$  or  $u_n$  is included in the minimand of relation (2. 9) or (2. 10) so that  $u_n$  and  $v_n$  in (2. 15) may satisfy (2. 13). The above model in which  $u_n$  and  $v_n$  satisfy (2. 13) is a generalisation of the GI/G/1 queueing process discussed by Lindley (1952) and Loynes (1962) and is investigated in Chapters 4 and 5.

Stochastic properties of the arrival process in the  $n$ th cycle,  $x_n(t)$  ( $0 < t \leq T$ ) are left unspecified as yet. Vehicles arriving at the intersection during the  $n$ th red period have to wait until the beginning of the  $n$ th green period at the shortest. Therefore if the distribution of  $x_n(r)$  is known, then that of  $x_n(t)$  ( $0 < t < r$ ) is unnecessary for the following analysis. Let  $\underline{E}(\cdot)$  denote the expectation of  $\cdot$ . To begin with, suppose that arrival processes in green periods satisfy the following condition:

- (E) The arrival processes in green periods  $x_n(t)$  ( $r \leq t \leq T$ ) are m.i.i.d. with  $\underline{E}\{x_n(T)\} = \lambda T (< \infty)$  and independent of departure headways.

Since  $x_n(t)$  ( $r \leq t \leq T$ ) is independent of random vectors  $(f_n, t_{n1}, \dots, t_{nf_n})$ , it follows from (2. 12) that for any integer  $k$ ,



$$(2.16) \quad \Pr\{v_n=k\} = \sum_{i,j} \Pr\{e_n=i\} \int \Pr\{\min\{x_n(r), x_n(\tau_{n1})-1, \\ \dots, x_n(\tau_{nj})-j, x_n(T)-i-j\}=k\} d\tau P\{f_n=j, t_{n1} \leq \tau_{n1}, \\ \dots, t_{nj} \leq \tau_{nj}\}.$$

Conditions (C) and (D) imply that the random vectors  $(f_n, t_{n1}, \dots, t_{nf_n})$  are m.i.i.d. and so are random variables  $e_n$ . Consequently, on account of Condition (A), the two-dimensional random vectors  $(u_n, v_n)$  defined by (2.11) and (2.12) are also m.i.i.d. Hence the queue length  $L_n$  at the beginning of the  $n$ th cycle satisfies model (2.15) in which the random vectors  $(u_n, v_n)$  are m.i.i.d. and  $u_n$  and  $v_n$  satisfy (2.13) and (2.14). This model has been discussed in Chapter 4. One of the results shown in Chapter 4 is:

Lemma 1.

Suppose that random vectors  $(u_n, v_n)$  ( $n=0,1,2,\dots$ ) are m.i.i.d. and  $u_n$  and  $v_n$  satisfy (2.13) and (2.14). Furthermore suppose that

$$(2.17) \quad \Pr\{v_n > 0\} > 0.$$

If and only if  $E\{u_n\} < 0$ , then the d.f. of  $L_n$  defined iteratively by (2.15) converges to a unique honest d.f. independent of any initial distribution of  $L_0$ , as  $n$  tends to infinity.

Denote by  $A*B$  the convolution of a d.f.  $A$  with a d.f.  $B$ . Clearly, by Condition (C),

$$(2.18) \quad H(g) = E\{f_n\} = \sum_{i=1}^{\infty} F_1 * \dots * F_i(g),$$

where  $H(\cdot)$  is the renewal function of the renewal sequence  $\{s_{ni}; i=1,2,\dots\}$  (see, Smith (1961)). Therefore by Conditions (D) and (E),

$$E\{u_n\} = \lambda T - H(g) - p.$$

This equation and Lemma 1 lead immediately to a necessary and sufficient condition under which the distribution of the queue length  $L_n$  at the beginning of the  $n$ th cycle converges to a unique stationary distribution independent of any initial distribution of  $L_0$ , as  $n$  tends to infinity. It is clear that if the queue length at the beginning of each cycle has a stationary distribution, then the queue length at an arbitrarily fixed time from the beginning of each cycle has a stationary distribution, too. This consequence is called briefly throughout this chapter: The traffic light queue has a stationary distribution.

#### Theorem 1.

Suppose that Conditions (A) through (E) and Condition (2.17) are satisfied. The fixed-cycle traffic light queue has a unique stationary distribution independent of any initial distribution, if and only if

$$(2.19) \quad \lambda T < H(g) + p,$$

where  $H(g)$  is given by (2.18).

Since by (2. 12)

$$v_n = \min\{x_n(r), x_n(t_{n1}) - 1, \dots, x_n(t_{nf_n}) - f_n, x_n(T) - e_n - f_n\},$$

Condition (2. 17) is not satisfied for a very light traffic such that  $x_n(T) \leq e_n + f_n$  a.s. For the stationary stream without after-effects of arriving vehicles, however, Condition (2. 17) is satisfied unless intensity  $\lambda$  vanishes. Condition (2. 17) is not satisfied also for an artificially controlled traffic which guarantees  $x_n(r) = 0$  a.s.; for instance, the ideal synchronization of sequential traffic lights may assure  $x_n(r) = 0$  a.s. This, however, is hindered by the variation of vehicles' velocities and incoming streams from upstream minor roads. Thus Condition (2. 17) is satisfied for a usual intersection controlled by a traffic light. Suppose temporarily that the conventional assumption on departure headways is satisfied: that is, for d.f.  $L$  and mean  $\ell$  of starting delays,  $s_{n1}$  is distributed with d.f.  $L * F$  with mean  $(\ell + m)$  and for  $i = 2, 3, \dots$ ,  $s_{ni}$  is distributed with d.f.  $F$  with mean  $m$ . Then, according to Smith (1958),

$$(2. 20) \quad H(g) = (g - \ell + E(\zeta)) / m - 1,$$

where  $\zeta$  is the residual useful life. Therefore, (2. 19) can be rewritten as

$$(2. 21) \quad \lambda m T < g - \ell + \{E(\zeta) - (1 - p)m\}.$$

In order to discuss the difference between (2. 21) and the result obtained in Chapter 4, in which the "preemptive resume" service discipline is assumed instead of Condition (D), let define the clearance time as the time from the end of the green period to the instant at which the last vehicle that fails to stop for the amber signal passes the stop line. Denote by  $c$  the mean clearance time. Then, by the definition,  $c = pE(\zeta)$ . Since the clearance time is thought of as effectively green, the result in Chapter 4 can be rewritten as

$$(2. 22) \quad \lambda mT < g - \ell + pE(\zeta).$$

Clearly, this right member is not less than that of (2. 21) and the difference between them is  $(1-p)\{m - E(\zeta)\}$ . Thus, if  $p=1$ , then (2. 22) agrees with (2. 21). Since  $p$  is actually near to one, (2. 21) fits the actual situations with Condition (D). Finally, Conditions (2. 21) and (2. 22) with  $p$  substituted by one include Darroch's condition (1964) as a simple case.

In the above, the arrival processes in green periods are assumed to be m.i.i.d. This assumption seems to be rather reasonable, especially for arrivals at an isolated signalized intersection. It is open to question, however, whether arrival processes in green

periods at a usual signalized intersection are mutually independent, for they are influenced by upstream and downstream bottlenecks, traffic lights of other intersections and so on. In the following part of this section, the arrival processes in green periods are permitted to be dependent. Suppose that the arrival processes in green periods satisfy the following condition instead of Condition (E):

- (F) The sequence of arrival processes in green periods  $\{x_n(t) \ (r \leq t \leq T); \ n=0,1,2,\dots\}$  is strictly stationary and independent of departure headways. In particular,  $\{x_n(T)\}$  is ergodic or metrically transitive (see Doob (1953), page 457) and
- $$E\{x_n(T)\} = \lambda T < \infty.$$

This condition is much weaker than Condition (E) and appears to be quite reasonable and practical. From the expression analogous to (2. 16), it follows that under Conditions (C), (D) and (F), the sequence of the random vectors  $\{(u_n, v_n)\}$  defined by (2. 11) and (2. 12) is strictly stationary. Moreover,  $\{u_n\}$  obeys the strong law of large numbers, because  $f_n$  and  $e_n$  are m.i.i.d. and  $\{x_n(T)\}$  is ergodic (Doob (1953), page 465). Hence the queue length  $L_n$  at the beginning of the  $n$ th cycle satisfies model (2. 15) in which  $\{(u_n, v_n)\}$  is strictly stationary,  $\{u_n\}$  obeys the strong

law of large numbers, and  $u_n$  and  $v_n$  satisfy (2. 13). This model has been discussed in Chapter 5. One of the results shown in Chapter 5 is:

Lemma 2.

Suppose that  $\{(u_n, v_n); n=0,1,2,\dots\}$  is strictly stationary and  $\{u_n\}$  obeys the strong law of large numbers and that  $u_n$  and  $v_n$  satisfy (2. 13). If  $E\{u_n\} < 0$ , then the d.f. of  $L_n$  defined iteratively by (2. 15) converges to a unique honest d.f. independent of any initial distribution of  $L_0$ , as  $n$  tends to infinity. From Conditions (C), (D) and (F) it follows that

$$E\{u_n\} = \lambda T - H(g) - p,$$

where  $H(g)$  is given by (2. 18). In exactly the same manner as Theorem 1, this equation and Lemma 2 lead directly to a sufficient condition under which the fixed-cycle traffic light queue has a stationary distribution.

Theorem 2.

Suppose that Conditions (A) through (D) and (F) are satisfied. The fixed-cycle traffic light queue has a unique stationary distribution independent of any initial distribution, if

$$(2. 23) \quad \lambda T < H(g) + p,$$

where  $H(g)$  is given by (2. 18).

Suppose that the conventional assumption on departure headways is satisfied. Then, in much the same way as the remark following Theorem 1, Condition (2. 23) reduces to the form of (2. 21). When  $p=1$ , this condition agrees with the result obtained in Chapter 5, in which, instead of Condition (D), the "preemptive resume" service discipline is assumed.

### 6. 3. Stationary queue length distribution

In this section, a fixed-cycle traffic light queue with regular departure headways depending upon positions is dealt with and its stationary queue length distribution is derived from familiar ways used by Bailey (1954), Newell (1960 a), Darroch (1964) and others. Suppose that Conditions (A), (B) and (D) are satisfied and that a departure headway of position  $i(i=1,2,\dots)$  is a constant positive number  $s_i$ . Clearly Condition (C) is satisfied. It can be assumed without loss of generality that  $s_1 < g$ . By (2. 1), the saturation flow  $f_n$  of the  $n$ th green period is a constant positive integer  $M$  such that  $\sum_{i=1}^M s_i \leq g < \sum_{i=1}^{M+1} s_i$ . For convenience of notations, put

$$(3. 1) \quad s_{M+1} = g - \sum_{i=1}^M s_i.$$

Suppose that the stream of vehicles in the straight-through lane arriving at the intersection is a stationary one without after-effects and has a finite and positive intensity  $\lambda$ . This assumption implies that Condition (E) is satisfied and  $x_n(t)$  ( $r \leq t \leq T$ ) have a probability generating function (p.g.f.) in the following form (see Khintchine (1960), page 34): for a complex variable  $z$ ,

$$(3. 2) \quad \exp\{\lambda t(\eta(z)-1)\},$$

where  $\eta(z)$  is an arbitrary p.g.f. given by



$$(3.3) \quad \eta(z) = \sum_{k=1}^{\infty} p_k z^k$$

and

$$(3.4) \quad \alpha \equiv \eta'(1) < \infty.$$

It is assumed that

$$(3.5) \quad p_1 > 0$$

and that for a sufficiently small positive number  $\delta$ , the radius of convergence of  $\eta(z)$  is greater than  $(1+\delta)$ , that is,

$$(3.6) \quad \limsup_{k \rightarrow \infty} p_k^{\frac{1}{k}} < 1/(1+\delta).$$

For instance, the Borel-Tanner distribution  $p_k$  ( $k=1,2,\dots$ ) (see Miller (1961 a)) given by

$$p_k = k^{k-1} \gamma^{k-1} e^{-\gamma k/k} !$$

satisfies (3.5) and (3.6) for positive parameter  $\gamma$  excluding the neighbourhood of 1, because the p.g.f. of this distribution has the radius of convergence  $e^{\gamma-1}/\gamma$ .

Moreover suppose that for the positive probability  $p$  in Condition (D),

$$(3.7) \quad \lambda \alpha T < M+p.$$

Then it follows from Theorem 1 that the distribution of the queue length  $L_n$  converges to a unique stationary distribution, as  $n$  tends to infinity.

Let  $\xi(z)$  be the p.g.f. of the random variable  $(1-e_n)$ ,

that is,

$$(3.8) \quad \xi(z) = p + qz,$$

where  $q=1-p$  and  $0 < p \leq 1$ . Denote by  $\tilde{L}_n(z)$ ,  $\tilde{N}_{nm}(z)$  and  $\tilde{y}_{nm}(z)$  the p.g.f.s for  $L_n$ ,  $N_{nm}$  and  $y_{nm}$  ( $n=0,1,2,\dots$ ,  $m=1,2,\dots,M+1$ ), respectively. Then from (2.6), (2.7) and (2.8) it follows that

$$\tilde{N}_{n0}(z) = \tilde{L}_n(z) \cdot \tilde{y}_{n0}(z),$$

for  $m=1,2,\dots,M$ ,

$$\tilde{N}_{nm}(z) = \frac{\tilde{y}_{nm}(z)}{z} \tilde{N}_{nm-1}(z) + (1 - \frac{\tilde{y}_{nm}(z)}{z}) \tilde{N}_{nm-1}(0)$$

and

$$\tilde{L}_{n+1}(z) = \frac{\xi(z) \tilde{y}_{nM+1}(z)}{z} \tilde{N}_{nM}(z) + (1 - \frac{\xi(z) \tilde{y}_{nM+1}(z)}{z}) \tilde{N}_{nM}(0).$$

Therefore

$$(3.9) \quad \begin{aligned} \tilde{L}_{n+1}(z) = & \xi(z) \tilde{y}_{n0}(z) \left( \prod_{m=1}^{M+1} \frac{\tilde{y}_{nm}(z)}{z} \right) \tilde{L}_n(z) \\ & + (1 - \frac{\xi(z) \tilde{y}_{nM+1}(z)}{z}) \tilde{N}_{nM}(0) \\ & + \xi(z) \sum_{m=0}^{M-1} (1 - \frac{\tilde{y}_{nm+1}(z)}{z}) \left( \prod_{j=m+2}^{M+1} \frac{\tilde{y}_{nj}(z)}{z} \right) \tilde{N}_{nm}(0). \end{aligned}$$

Since from Theorem 1, as  $n$  tends to infinity,  $\tilde{L}_n(z)$  and  $\tilde{N}_{nm}(z)$  ( $m=0,1,\dots,M$ ) converge to the p.g.f.s for the stationary distributions, say  $\tilde{L}(z)$  and  $\tilde{N}_m(z)$  ( $m=0,1,\dots,M$ ), letting  $n$  tend to infinity in (3.9) yields

$$(3.10) \quad (z^{M+1} - \xi(z) \prod_{m=0}^{M+1} \tilde{y}_m(z)) \tilde{L}(z) = (z^{M+1} - z^M \xi(z) \tilde{y}_{M+1}(z)) \tilde{N}_M(0)$$

$$+\xi(z)\sum_{n=0}^{M-1}(z-\tilde{y}_{m+1}(z))(\prod_{j=m+2}^{M+1}\tilde{y}_j(z))z^{m\tilde{N}_m(0)},$$

where by (2. 4), (2. 5), (3. 1) and (3. 2),

$$(3. 11) \quad \tilde{y}_0(z) \equiv \tilde{y}_{n0}(z) = \exp\{\lambda r(\eta(z)-1)\}$$

and for  $m=1, 2, \dots, M+1$

$$(3. 12) \quad \tilde{y}_m(z) \equiv \tilde{y}_{nm}(z) = \exp\{\lambda s_m(\eta(z)-1)\}.$$

Put  $f(z)$  and  $g_m(z)$  ( $m=0, 1, \dots, M$ ) as follows and rewrite them by use of (3. 1), (3. 8), (3. 11) and (3. 12):

$$(3. 13) \quad f(z) \equiv z^{M+1} - \xi(z) \prod_{m=0}^{M+1} \tilde{y}_m(z) = z^{M+1} - (p+qz) \exp\{\lambda T(\eta(z)-1)\},$$

for  $m=0, 1, \dots, M-1$ ,

$$(3. 14) \quad g_m(z) \equiv z^m \xi(z) (z - \tilde{y}_{m+1}(z)) (\prod_{j=m+2}^{M+1} \tilde{y}_j(z)) \\ = z^m (p+qz) [z - \exp\{\lambda s_{m+1}(\eta(z)-1)\}] \exp\{\lambda \sum_{j=m+2}^{M+1} s_j \\ (\eta(z)-1)\}$$

and

$$(3. 15) \quad g_M(z) \equiv z^{M+1} - z^M \xi(z) \tilde{y}_{M+1}(z) = z^{M+1} - z^M (p+qz) \cdot \\ \exp\{\lambda s_{M+1}(\eta(z)-1)\}.$$

Then Equation (3. 10) reduces to

$$(3. 16) \quad \tilde{L}(z) = \sum_{m=0}^M g_m(z) \tilde{N}_m(0) / f(z).$$

It should be noted that  $\tilde{L}(z)$  in (3. 16) contains unknowns  $\tilde{N}_m(0)$ . These unknowns can be determined by a system of linear equations in the sequel.

Since  $\tilde{L}(z)$  is a p.g.f., it is analytic inside the unit circle and by Abel's theorem,  $\lim_{z \rightarrow 1-0} \tilde{L}(z) = 1$ . Therefore, by (3. 4) and (3. 13) through (3. 16),

$$(3. 17) \quad M+p-\lambda\alpha T = (p-\lambda\alpha s_{M+1})\tilde{N}_M(0) + \sum_{m=0}^{M-1} (1-\lambda\alpha s_{m+1})\tilde{N}_m(0).$$

Moreover the numerator of (3. 16) must have the same zeros inside the unit circle as the denominator  $f(z)$ .

### Lemma 3.

Suppose that Assumptions (3. 5) through (3. 7) are satisfied. Then the denominator  $f(z)$  has  $(M+1)$  distinct zeros, say,  $z_1, z_2, \dots, z_{M+1}$  on and inside the unit circle, where  $z_1 = 1$  and for  $k=2, 3, \dots, M+1$ ,  $0 < |z_k| < 1$ .

Proof.

Put  $f_1(z) = z^{M+1}$  and  $f_2(z) = (p+qz)\exp\{\lambda T(\eta(z)-1)\}$ . Assumption (3. 6) implies that  $f_2(z)$  is analytic within the circle  $|z| = 1+\delta$ . Clearly  $f_1(1) = f_2(1)$  and by (3. 3),  $|f_2(e^{i\theta})| = \sqrt{1-2pq(1-\cos\theta)} \exp\{-\lambda T(1 - \sum_{k=1}^{\infty} p_k \cos k\theta)\}$ . From Assumption (3. 5), for  $\theta \neq 2k\pi$  ( $k=0, \pm 1, \dots$ ),  $\sum_{k=1}^{\infty} p_k \cos k\theta < 1$ . Hence for  $\theta \neq 2k\pi$ ,  $|f_2(e^{i\theta})| < 1 = f_1(e^{i\theta})$ , that is,  $|f(e^{i\theta})| = |f_1(e^{i\theta}) - f_2(e^{i\theta})| > 0$ . Since  $|f_2'(e^{i\theta})| \leq q + \lambda\alpha T$ , Assumption (3. 7) implies that for all  $\theta$ ,  $|f_2'(e^{i\theta})| < |f_1'(e^{i\theta})|$ .

Therefore, for the sufficiently small positive number  $\delta$ ,  $|f_1(z)| > |f_2(z)|$  on the circle  $|z| = 1+\delta$ . Clearly,  $f_1(z)$  has  $(M+1)$  zeros inside the circle  $|z| = 1+\delta$  and by Rouché's theorem (see Titchmarsh (1939), page 116),

$f(z)=f_1(z)-f_2(z)$  also has  $(M+1)$  zeros, say  $z_k$  ( $k=1,2,\dots,M+1$ ), inside the circle  $|z|=1+\delta$ . Moreover,  $|f'(z_k)| = |z_k^M| |M+p/\xi(z_k)-\lambda T z_k \eta'(z_k)| \geq |z_k^M| (M+p-\lambda \alpha T) > 0$ . Hence, zeros  $z_k$  are of the first order and distinct. The proof is concluded.

From this lemma and (3. 16) it follows that for  $k=2,3,\dots,M+1$ ,

$$(3. 18) \quad \sum_{m=0}^M g_m(z_k) \tilde{N}_m(0)=0.$$

Define an  $(M+1) \times (M+1)$  matrix  $\underline{A}=(a_{k\ell})$  and  $(M+1)$  dimensional vectors  $\underline{x}=(x_\ell)$  and  $\underline{b}=(b_\ell)$  as follows:

$$\begin{aligned} a_{1\ell} &= 1-\lambda \alpha s_\ell \text{ for } \ell=1,2,\dots,M, a_{1M+1}=p-\lambda \alpha s_{M+1}, \\ a_{k\ell} &= g_{\ell-1}(z_k) \text{ for } \ell=1,2,\dots,M+1 \text{ and } k=2,3,\dots,M+1, \\ x_\ell &= \tilde{N}_{\ell-1}(0) \text{ for } \ell=1,2,\dots,M+1, b_1=M+p-\lambda \alpha T \text{ and} \\ b_\ell &= 0 \text{ for } \ell=2,3,\dots,M+1. \end{aligned}$$

Then the system of  $(M+1)$  linear equations (3. 17) and (3. 18) can be rewritten in the matrix form:

$$\underline{A}\underline{x}=\underline{b}.$$

This system of linear equations must have a unique solution and the matrix  $\underline{A}$  is nonsingular, because from Theorem 1,  $\tilde{L}(z)$  is unique. Therefore, for  $m=0,1,\dots,M$ ,

$$(3. 19) \quad \tilde{N}_m(0)=(M+p-\lambda \alpha T) \underline{A}_{1m+1}/|\underline{A}|,$$

where  $\underline{A}_{1m+1}$  is the cofactor of the element  $a_{1m+1}$ . Hence

combination of Equations (3. 16) and (3. 19) leads to

$$(3. 20) \quad \tilde{L}(z) = (M+p-\lambda\alpha T) \sum_{m=0}^M A_{1m+1} s_m(z) / (|A| f'(z)).$$

If  $s_1=s_2=\dots=s_{M+1}$  and  $p=1$ , then this p.g.f. will agree with the Darroach's result (1964), though he made less practical assumptions concerning on (3. 2) than Assumptions (3.5) and (3. 6).

Differentiating (3. 20) yields

$$(3. 21) \quad \tilde{L}'(1) = \left\{ (M+p-\lambda\alpha T) \sum_{m=0}^M A_{1m+1} s_m''(1) - |A| f''(1) \right\} / (2|A| f'(1)).$$

While by (3. 13) through (3. 15),

$$(3. 22) \quad f'(1) = M+p-\lambda\alpha T,$$

$$(3. 23) \quad f''(1) = M(M+1) - \lambda T(2q\alpha + \lambda T\alpha^2 + \beta),$$

for  $m=0, 1, \dots, M-1$

$$(3. 24) \quad s_m''(1) = 2(1-\lambda\alpha s_{m+1})(m+q+\lambda\alpha \sum_{j=m+2}^{M+1} s_j) - \lambda s_{m+1}(\lambda s_{m+1}\alpha^2 + \beta)$$

and

$$(3. 25) \quad s_M''(1) = 2(1-\lambda\alpha s_{M+1})(M+q) - 2(M+1)q - \lambda s_{M+1}(\lambda s_{M+1}\alpha^2 + \beta),$$

where  $\beta = \eta''(1)$ . Therefore from (3. 21) through (3. 25) it follows that the expectation of the stationary queue length at the beginning of the cycle,  $E\{L\}$  is given by:

$$\begin{aligned}
(3.26) \quad \underline{E}\{L\} = & \left\{ \sum_{m=1}^{M+1} A_{1m} [(1-\lambda \alpha s_m)(m-p) - \lambda s_m (\lambda s_m \alpha^2 + \beta)/2] \right. \\
& + \sum_{m=1}^M \lambda \alpha (1-\lambda \alpha s_m) A_{1m} \sum_{j=m+1}^{M+1} s_j - q(M+1) A_{1M+1} \left. \right\} / |A| \\
& - \{M(M+1) + \lambda T(2q\alpha + \lambda T\alpha^2 + \beta)\} / 2(M+p-\lambda \alpha T).
\end{aligned}$$

Let be concerned with the expected delay per vehicle in a stationary state, say  $\underline{E}\{W\}$ . Denote by  $N(t)$  the queue length at time  $t$  from the beginning of the cycle in a stationary state. Then

$$(3.27) \quad \underline{E}\{W\} = \underline{E}\left\{\int_0^T N(t) dt\right\} / (\lambda \alpha T).$$

While by Fubini theorem (see Loève (1963), page 136) and (2.3),

$$\begin{aligned}
(3.28) \quad \underline{E}\left\{\int_0^T N(t) dt\right\} = & \int_0^r \underline{E}N(t) dt + \sum_{m=1}^{M+1} \int_{t_{nm-1}}^{t_{nm}} \underline{E}N(t) dt \\
= & r \underline{E}\{L\} + \lambda \alpha r^2 / 2 + \sum_{m=1}^{M+1} \left\{ s_m \tilde{N}'_{m-1}(1) + \lambda \alpha s_m^2 (1 - \tilde{N}_{m-1}(0)) / 2 \right\}.
\end{aligned}$$

Moreover,

$$\tilde{N}'_0(1) = \underline{E}\{L\} + \lambda \alpha r$$

and for  $m=1, 2, \dots, M$ ,

$$\tilde{N}'_m(1) = \underline{E}\{L\} + \lambda \alpha r - \sum_{j=1}^m (1 - \tilde{N}_{j-1}(0)) (1 - \lambda \alpha s_j).$$

Consequently from (3.19), (3.27) and (3.28),

$$\begin{aligned}
(3.29) \quad \underline{E}\{W\} = & \underline{E}\{L\} / (\lambda \alpha) + (M+p-\lambda \alpha T) \left[ \sum_{m=1}^M A_{1m} \left\{ (1-\lambda \alpha s_m) \sum_{j=m+1}^{M+1} s_j \right. \right. \\
& \left. \left. - \lambda \alpha s_m^2 / 2 \right\} - \lambda \alpha s_{M+1}^2 A_{1M+1} / 2 \right] / \lambda \alpha T |A| + \{rg + (r^2
\end{aligned}$$

$$+ \sum_{m=1}^{M+1} s_m^2)/2\}/T - \sum_{m=1}^M (1 - \lambda \alpha s_m) \sum_{j=m+1}^{M+1} s_j / (\lambda \alpha T).$$

In the above, arrivals are assumed to be stationary. Now suppose that arrivals are nonstationary; that is, the p.g.f. of  $x_n(t)$  is assumed to be  $\exp\{\lambda(t)(\gamma(z)-1)\}$ , instead of (3. 2). For these generalized arrivals,  $\tilde{L}(z)$ ,  $\underline{E}\{L\}$  and  $\underline{E}\{w\}$  can be obtained by much the same way as in the above.



#### 6. 4. Semi-vehicle-actuated traffic light queue

Consider a queue of vehicles in a straight-through lane of a major road leading into an intersection controlled by a semi-vehicle-actuated traffic light with detectors on a minor road. Denote by  $r_n$ ,  $g_n$  and  $T_n$  ( $n=0,1,2,\dots$ ) the lengths of the  $n$ th red period, the  $n$ th green period and the  $n$ th cycle, respectively. Stochastic properties of these random variables are determined by the control algorithm of the traffic light in front of the queue, stochastic properties of the stream of vehicles arriving at the intersection along the minor road, those of their departure headways, and so on. What controls the queue, however, is not these many factors but random variables  $r_n$  and  $g_n$ . Therefore, in order to maintain generality of the following analysis, suppose that:

- (A') Positive random vectors  $(r_n, g_n)$  ( $n=0,1,2,\dots$ ) are m.i.i.d. and  $r_n$  and  $g_n$  have finite means  $r$  and  $g$ , respectively. In particular,  $g_n$  are a.s. bounded above.

Consequently  $T_n$  have finite mean  $T(=r+g)$ . This condition includes Condition (A) as a particular case, but the latter half of this assumption may not be satisfied for some semi-vehicle-actuated traffic lights. Moreover suppose that Conditions (B) through (D) in Section 6. 2

are satisfied and that departure headways are independent of  $r_n$  and  $g_n$ . Let the 0th cycle begin at time zero.

Denote by  $x(t)$  ( $t > 0$ ) the number of vehicles in the straight-through lane of the major road arriving at the intersection in time interval  $(0, t]$ . This process is called the arrival process on the major road in this section. Because the arrival processes in green periods used in the preceding sections are directly affected by random variables  $r_n$  and  $g_n$ . Suppose that

- (G) The arrival process on the major road  $x(t)$  ( $t > 0$ ) is a stationary stream without after-effects, has finite intensity  $\lambda(>0)$  and independent of  $r_n$ ,  $g_n$  and departure headways.

It is to be noted that this condition is identical with the condition that the arrival process has stationary independent increments with finite mean (see Doob (1953), page 96). The notation similar to that in Section 6. 2 is used in this section. Define the saturation flow  $f_n$  ( $n=0,1,2,\dots$ ) and the random time  $t_{nm}$  ( $m=0,1,2,\dots$ ) by (2. 1) and (2. 3) with  $r$ ,  $g$  and  $T$  replaced by  $r_n$ ,  $g_n$  and  $T_n$ . That is,

$$(4. 1) \quad f_n = \sup \left\{ k; \sum_{i=0}^k s_{ni} \leq g_n \right\}$$

and

$$(4. 2) \quad t_{nm} = \inf \left\{ r_n + \sum_{i=0}^m s_{ni}, T_n \right\}.$$

By Condition (A') and (C), saturation flows  $f_n$  are m.i.i.d. and satisfy (2. 2), and for fixed  $m$ , random times  $t_{nm}$  are also m.i.i.d. Since clearly,

$$(4. 3) \quad x_n(t_{nm}) = x\left(\sum_{i=0}^{n-1} T_i + t_{nm}\right) - x\left(\sum_{i=0}^{n-1} T_i\right),$$

for fixed  $m$ , random variables  $x_n(t_{nm})$  are m.i.i.d. In the same way as in (2. 10), it follows from Conditions (A'), (B) and (D) that the following recurrence relation between  $L_n$  and  $L_{n+1}$  ( $n=0,1,2,\dots$ ) holds:

$$(4. 4) \quad L_{n+1} = L_n + x_n(T_n) - e_n - f_n, \\ \text{if } L_n + \min_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T_n) - e_n - f_n \} > 0, \\ = 0, \text{ otherwise.}$$

Put for  $n=0,1,2,\dots$ ,

$$(4. 5) \quad u_n = x_n(T_n) - e_n - f_n$$

and

$$(4. 6) \quad v_n = \min_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T_n) - e_n - f_n \}.$$

Then by Condition (D) and (2. 2),  $u_n$  and  $v_n$  satisfy (2. 13) and (2. 14). Since by (2. 2) and Conditions (A') and (G) (see Khintchine (1960), page 34),  $\Pr\{x_n(r_n) > K+1\} > 0$ ,  $v_n$  satisfies Condition (2. 17). Moreover it follows from (4. 1) through (4. 3) that for  $m < n$  and arbitrary nonnegative integers  $k, \ell$ ,

$$\begin{aligned}
\Pr\{v_m=k, v_n=l\} &= \int \Pr\{\min\{x(y+z_m)-x(y), x(y+z_m+t_{m1}) \\
&\quad -x(y)-1, \dots, x(y+z_m+t_{mf_m})-x(y)-f_m, x(y+z_m+\theta_m)-x(y)-e_m-f_m\} \\
&\quad =k, \min\{x(y+z_m+\theta_m+z+z_n)-x(y+z_m+\theta_m+z), x(y+z_m+\theta_m+z+z_n+t_{n1}) \\
&\quad -x(y+z_m+\theta_m+z)-1, \dots, x(y+z_m+\theta_m+z+z_n+t_{nf_n})-x(y+z_m+\theta_m+z) \\
&\quad -f_n, x(y+z_m+\theta_m+z+z_n+\theta_n)-x(y+z_m+\theta_m+z)-e_n-f_n\}=l\} dP\left\{\sum_{i=1}^{m-1} T_i \right. \\
&\quad \left. \leq y, r_m \leq z_m, g_m \leq \theta_m, \sum_{i=m+1}^{n-1} T_i \leq z, r_n \leq z_n, g_n \leq \theta_n\right\} \\
&= \iiint \Pr\{\min\{x(z_m), x(z_m+t_{m1})-1, \dots, x(z_m+t_{mf_m})-f_m, \\
&\quad x(z_m+\theta_m)-e_m-f_m\}=k\} \Pr\{\min\{x(z_n), x(z_n+t_{n1})-1, \dots, \\
&\quad x(z_n+t_{nf_n})-f_n, x(z_n+\theta_n)-e_n-f_n\}=l\} dP\{r_m \leq z_m, g_m \leq \theta_m\} dP\{r_n \\
&\quad \leq z_n, g_n \leq \theta_n\} = \iiint \Pr\{\min\{x(y+z_m)-x(y), x(y+z_m+t_{m1}) \\
&\quad -x(y)-1, \dots, x(y+z_m+t_{mf_m})-x(y)-f_m, x(y+z_m+\theta_m)-x(y) \\
&\quad -e_m-f_m\}=k\} dP\left\{\sum_{i=0}^{m-1} T_i \leq Y\right\} dP\{r_m \leq z_m, g_m \leq \theta_m\} \iiint \Pr\{\min\{x(z+z_n) \\
&\quad -x(z), x(z+z_n+t_{n1})-x(z)-1, \dots, x(z+z_n+t_{nf_n})-x(z)-f_n, \\
&\quad x(z+z_n+\theta_n)-x(z)-e_n-f_n\}=l\} dP\left\{\sum_{i=0}^{n-1} T_i \leq z\right\} dP\{r_n \leq z_n, g_n \leq \theta_n\} \\
&= \Pr\{v_m=k\} \Pr\{v_n=l\}.
\end{aligned}$$

In a similar way it can be shown that the random vectors  $(u_n, v_n)$  are m.i.i.d. Hence from (4. 4) through (4. 6) it follows that the queue length  $L_n$  at the beginning of the  $n$ th cycle satisfies model (2. 15) in which the random

vectors  $(u_n, v_n)$  are m.i.i.d. and  $u_n$  and  $v_n$  satisfy (2. 13), (2. 14) and Condition (2. 17). Since

$$\underline{E}(u_n) = \lambda T - \underline{E}\{H(g_n)\} - p,$$

Lemma 1 leads directly to:

Theorem 3.

Suppose that Conditions (A') through (D) and (G) are satisfied. The semi-vehicle-actuated traffic light queue has a unique limiting distribution independent of any initial distribution, if and only if

$$(4. 7) \quad \lambda T < \underline{E}\{H(g_n)\} + p,$$

where  $H(\cdot)$  is the renewal function of departure headways and given by (2. 18).

Suppose temporarily that the conventional assumption on departure headways is satisfied. Let  $\ell_n$  and  $\zeta_n$  ( $n=0,1,2,\dots$ ) be the starting delay and the residual useful life in the  $n$ th cycle. Then in the same way as (2. 20),

$$(4. 8) \quad \underline{E}\{H(g_n)\} = \underline{E}\{[g_n + \zeta_n - \ell_n]^+\} / m - 1,$$

where  $[\cdot]^+ = \max\{\cdot, 0\}$ . Note that it can not be always assumed that  $\ell_n \leq g_n + \zeta_n$  a.s. It follows from (4. 8) that (4. 7) reduces to

$$(4. 9) \quad \lambda m T < \underline{E}\{[g_n + \zeta_n - \ell_n]^+\} - (1-p)m.$$

If

$$(4. 10) \quad p = 1 - \underline{E}\{[g_n + \zeta_n - \ell_n]^+ - [g_n - \ell_n]^+\} / m,$$

then (4. 9) reduces to

$$(4. 11) \quad \lambda mT < E\{[g_n - l_n]^+\}.$$

Now suppose that the following Conditions (A'') and (H) are satisfied instead of Conditions (A') and (G):

(A'') The sequence of positive random vectors  $\{(r_n, g_n); n=0, 1, 2, \dots\}$  is strictly stationary and ergodic, and  $E(r_n) = r < \infty$  and  $E(g_n) = g < \infty$ .

Denote by  $x_k$  ( $k=0, 1, 2, \dots$ ) the number of vehicles in the straight-through lane on the major road arriving at the intersection in the  $k$ th unit time interval  $(k, k+1]$ .

(H) The arrival process on the major road  $x(t)$  ( $t > 0$ ) is a stationary stream with finite intensity  $\lambda$  and independent of  $\{(r_n, g_n)\}$  and departure headways. In particular, the sequence  $\{x_k; k=0, 1, 2, \dots\}$  is ergodic.

Clearly these conditions include Conditions (A') and (G) as particular cases. Conditions (A'') and (C) imply that the sequence of saturation flows  $f_n$  defined by (4. 1) is strictly stationary. Besides, for arbitrary integers  $m_i, k_i$  ( $i=1, 2, \dots, i_0$ ) and  $n_j, l_j$  ( $j=1, 2, \dots, j_0$ ),

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \Pr\{f_{m_i} = k_i, f_{n_j+h} = l_j, i=1, \dots, i_0, j=1, \dots, j_0\} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \Pr\left\{\sup\{k; \sum_{i=1}^k s_{m_i} \leq x_{m_i}\} = k_i, i=1, \dots, i_0\right\} \Pr\left\{\sup\{k; \sum_{j=1}^k s_{n_j} \leq x_{n_j}\} = l_j, j=1, \dots, j_0\right\} \end{aligned}$$

$$\sum_{i=1}^k s_{n_j i} \leq y_j \} = l_j, \quad j=1, \dots, j_0 \} dP \{ s_{m_i} \leq x_i, s_{n_j+h} \leq y_j, \\ i=1, \dots, i_0, \quad j=1, \dots, j_0 \}.$$

Since according to the necessary and sufficient condition for ergodicity (see Billingsley (1965), page 17),

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \Pr \{ s_{m_i} \leq x_i, s_{n_j+h} \leq y_j, \quad i=1, \dots, i_0, \quad j=1, \dots, j_0 \} \\ = \Pr \{ s_{m_i} \leq x_i, \quad i=1, \dots, i_0 \} \Pr \{ s_{n_j} \leq y_j, \quad j=1, \dots, j_0 \},$$

it follows from Helly-Bray theorem (see Loève (1963), page 182) that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{h=0}^{n-1} \Pr \{ f_{m_i} = k_i, \quad f_{n_j+h} = l_j, \quad i=1, \dots, i_0, \\ j=1, \dots, j_0 \} \\ = \Pr \{ f_{m_i} = k_i, \quad i=1, \dots, i_0 \} \Pr \{ f_{n_j} = l_j, \quad j=1, \dots, j_0 \}.$$

That is,  $\{f_n: n=0, 1, 2, \dots\}$  is ergodic, and obeys the strong law of large numbers (see, Doob (1953), page 465).

It should be noted, however, that  $f_n$  does not satisfy (2. 2). Therefore Recurrence Relation (4. 4) between  $L_n$  and  $L_{n+1}$  ( $n=0, 1, 2, \dots$ ) must be rewritten as follows:

$$(4. 12) \quad L_{n+1} = L_n + x_n(T_n) - e_n - f_n, \\ \text{if } L_n + \inf_{0 \leq m \leq f_n} \{ x_n(t_{nm}) - m \}, x_n(T_n) - e_n - f_n \} > 0, \\ = 0, \text{ otherwise.}$$

Put instead of (4. 6),

$$(4.13) \quad v_n = \inf \left\{ \inf_{0 \leq m \leq f_n} \{x_n(t_{nm}) - m\}, x_n(T_n) - e_n - f_n \right\}.$$

Then,  $u_n$  defined by (4.5) and  $v_n$  satisfy (2.13), and not (2.14). For arbitrary integers  $n_j$  and  $k_j$  ( $j=1, \dots, j_0$ ), by (4.3) and (4.13),

$$\begin{aligned} \Pr\{v_{n_j} = k_j, j=1, \dots, j_0\} &= \Pr\left\{\inf\left\{\inf_{0 \leq t_{n_j m} \leq f_{n_j}} [x_{n_j}(t_{n_j m}) - m], x_{n_j}(T_{n_j}) - e_{n_j} - f_{n_j}\right\} = k_j, j=1, \dots, j_0\right\} \\ &= \int \Pr\left\{\inf\left\{\inf_{0 \leq t_{n_j m} \leq f_{n_j}} \left[x\left(\sum_{i=0}^{n_j-1} (\gamma_i + \theta_i) + t_{n_j m}\right) - x\left(\sum_{i=0}^{n_j-1} \gamma_i + \theta_i\right) - m\right], \right. \right. \\ &\quad \left. \left. x\left(\sum_{i=0}^{n_j} \gamma_i + \theta_i\right) - x\left(\sum_{i=0}^{n_j-1} \gamma_i + \theta_i\right) - e_{n_j} - f_{n_j}\right\} = k_j, \right. \\ &\quad \left. j=1, \dots, j_0 \mid r_i \leq \gamma_i, \varepsilon_i \leq \theta_i, i=0, 1, \dots, n_{j_0}\right\} \\ &\quad dP\{r_i \leq \gamma_i, \varepsilon_i \leq \theta_i, i=0, 1, \dots, n_{j_0}\}. \end{aligned}$$

Consequently from Conditions (A"), (C), (D) and (H), for any positive integer  $h$ ,

$$\Pr\{v_{n_j} = k_j, j=1, \dots, j_0\} = \Pr\{v_{n_j+h} = k_j, j=1, \dots, j_0\}.$$

In a similar way it can be shown that  $\{(u_n, v_n)\}$  is strictly stationary. Since  $\{e_n\}$  and  $\{f_n\}$  obey the strong law of large numbers, it suffices to prove that  $\{x_n(T_n)\}$  obeys this law, in order to prove  $\{u_n\}$  obeys



the law. Let  $y_n = [\sum_{i=0}^{n-1} T_i]$ , where  $[\cdot]$  means the maximal integer not exceeding  $\cdot$ . By Condition (A''),  $\lim_{n \rightarrow \infty} y_n/n = \lim_{n \rightarrow \infty} \left\{ \sum_{i=0}^{n-1} T_i + (y_n - \sum_{i=0}^{n-1} T_i) \right\} / n = \tilde{E}(T_i) = T$  a.s. Therefore by Condition (H) and (4. 3),

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} x_i(T_i) &= \lim_{n \rightarrow \infty} \frac{1}{n} x\left(\sum_{i=0}^{n-1} T_i\right) \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{y_n}{n} \cdot \frac{1}{y_n} \sum_{k=0}^{y_n-1} x_k + \frac{1}{n} \left( x\left(\sum_{i=0}^{n-1} T_i\right) - x(y_n) \right) \right\} \\ &= \lambda T = \tilde{E}\{x_n(T_n)\} \text{ a.s.} \end{aligned}$$

Hence from (4. 5), (4. 12) and (4. 13) it follows that the queue length  $L_n$  at the beginning of the  $n$ th cycle satisfies model (2. 15) in which  $\{(u_n, v_n)\}$  is strictly stationary,  $\{u_n\}$  obeys the strong law of large numbers and  $u_n$  and  $v_n$  satisfy (2. 13). The following theorem results immediately from Lemma 2:

Theorem 4.

Suppose that Conditions (A'') through (D) and (H) are satisfied. The semi-vehicle-actuated traffic light queue has a unique limiting distribution independent of any initial distribution, if

$$(4. 14) \quad \lambda T < \tilde{E}\{H(g_n)\} + p,$$

where  $H(\cdot)$  is the renewal function of departure headways and is given by (2. 13).

This theorem gives a sufficient condition under which

the quite general semi-vehicle-actuated traffic light queue has a limiting distribution. If the conventional assumption on departure headways is satisfied, then it follows from (4. 3) and (4. 10) that Condition (4. 14) reduces to the form of (4. 11). This condition agrees with the result obtained in Chapter 5.

To maintain generality of the control algorithm of the traffic light, the arrival process on the minor road and so on, only one Condition (A') or (A'') has been made for them. As an example which satisfies Conditions (A') or (A''), consider a queue of vehicles in front of a pedestrian crossing controlled by a traffic light with detectors or push-buttons for arriving pedestrians. The control algorithm is:

- (I) The traffic light stays red for the fixed length  $r$  and turns green. It turns red at fixed time  $a$  ( $\geq 0$ ) after any pedestrian arrives at the pedestrian crossing.

Moreover the following reasonable condition is made:

- (J) Pedestrians arrive in groups at the pedestrian crossing. The interarrival times of successive groups are m.i.i.d.r.v.s with d.f.  $G$  and positive mean  $\mu$ . The queue of pedestrians discharges a.s. during one red period.

Suppose that the first group of pedestrians arrives at

time  $(-a)$ . Then the 0th cycle begins at time zero and each green period is composed of one residual useful life and one time interval of fixed length  $a$ . This implies that Condition (A'') is satisfied and that if  $G(Z)=1$  for some finite  $Z$ , then Condition (A') is also satisfied. Denote by  $R(\cdot)$  the renewal function of interarrival times: that is, for  $x \geq 0$ ,

$$(4.15) \quad R(x) = \sum_{k=1}^{\infty} G_k(x),$$

where  $G_k(x)$  is the  $k$ -fold convolution of  $G(x)$  with itself. Clearly,

$$(4.16) \quad E(T_n) = \mu \{1 + R(a+r)\}.$$

Moreover, for  $x < a$ ,  $\Pr\{g_n \leq x\} = 0$  and for  $x \geq a$ ,

$$\begin{aligned} \Pr\{g_n \leq x\} &= G(r+x) - G(a+r) + \sum_{k=1}^{\infty} \int_0^{a+r} \{G(r+x-y) \\ &\quad - G(a+r-y)\} dG_k(y) \\ &= G(r+x) + \int_0^{a+r} G(r+x-y) dR(y) - \{G(a+r) \\ &\quad + G * R(a+r)\}. \end{aligned}$$

Consequently,

$$(4.17) \quad E\{H(g_n)\} = \int_a^{\infty} H(x) dG(r+x) + \int_0^{a+r} \int_a^{\infty} H(x) dG(r+x-y) dR(y).$$

From Theorems 3 and 4, (4.16) and (4.17) it follows that the queue of vehicles in front of the pedestrian crossing has a unique limiting distribution if

$$\lambda\mu\{1+R(a+r)\} < p + \int_a^\infty H(x) dG(r+x) + \int_0^{a+r} \int_a^\infty H(x) dG(r+x - y) dR(y)$$

and, under the additional condition  $G(Z)=1$  for some finite  $Z$ , only if the above condition is satisfied, where  $H(x)$  and  $R(x)$  are given by (2. 18) and (4. 15), respectively.

#### 6. 5. Concluding remarks

The queue of vehicles in the straight-through lane at the intersection controlled by fixed-cycle or semi-vehicle-actuated traffic lights is discussed throughout this chapter. If  $g$  or  $g_n$  is interpreted as the length of the  $n$ th effective green period and if departure headways of vehicles turning right or left satisfy Condition (C), then all results of this paper remain valid for the queue of vehicles turning right or left.

With most vehicle-actuated traffic lights, maximum vehicle-extension periods are predetermined in order to prevent vehicles on intersecting roads from waiting indefinitely. Consequently, vehicle-actuated traffic lights in effect become fixed-cycle ones, if the traffic is fairly heavy on all phases. Thus, sufficient conditions derived in Section 6. 2 remain valid for the vehicle-actuated traffic light queue, if  $g$  and  $T$  are interpreted as lengths of the maximum green period and the maximum cycle.

Consider an intersection with  $J$  approaches which is controlled by a fixed-cycle traffic light. Suppose that vehicles arriving on all approaches satisfy assumptions in Theorem 1. Then from Theorem 1, the queue of vehicles on the  $j$ th approach has a stationary distribution, if and only if

$$H_j(g_j) + p_j - \lambda_j T > 0,$$

where suffix  $j$  refers to vehicles on the  $j$ th approach.

Hence if and only if

$$\min_{j=1, \dots, J} \{H_j(g_j) + p_j - \lambda_j T\} > 0,$$

then all queues at the intersection have stationary distributions; that is, the intersection is undersaturated.

In Theorems 2 and 4, departure headways are assumed to be a.s. greater than a positive number. This assumption is not essential and may be dropped.

## Chapter 7. Optimal Signal Settings

### 7. 1. Introduction

The optimal control of traffic lights is one of the main problems in the theory of traffic flow and many authors have dealt with this problem. Wardrop (1952), Webster (1958), Miller (1963 b), Rangarajan and Oliver (1967) and Allsop (1971 a, b) have discussed optimal signal settings for a fixed-cycle traffic light that minimize the mean delay per vehicle, or the mean overall delay per unit time. Dunne and Potts (1964, 1967), Green and Hartley (1966), Neimark and Fedotokin (1966), Grafton and Newell (1967), Green (1967), Martin-Löf (1967), Fedotokin (1969) and Gordon (1969) have investigated optimal control algorithms based upon the above mentioned or other criteria for a vehicle-actuated traffic light. Gazis (1964, 1965), Morgan and Little (1964), Little (1966), Stoffers (1968) and Gordon (1969) have dealt with the optimal control of a system of traffic lights based upon various criteria.

In this chapter, optimal signal settings based upon the criterion of minimum overall delay and that of minimizing a degree of saturation (of the whole intersection) are discussed for the fixed-cycle traffic light, where the

latter criterion is new and the degree of saturation is defined in Section 7. 2. In Section 7. 2, criteria for undersaturation of the whole intersection are derived from the results obtained in Chapter 4. These criteria for undersaturation may be important in design of the intersection in themselves. On the other hand, the optimal signal setting based upon the former criterion has been investigated by several authors, as noted in the above. In particular, Webster (1958) has developed the method of optimal signal setting which is popularly used. They, however, take little account of streams turning right, assuming that vehicles keep to the left side of the road. In this chapter, all streams that pass through the intersection are taken into consideration. Thus, the optimal signal setting must be formulated in terms of not mean effective green times but (real) green times, because mean effective green times of streams turning right depend on the signal setting. In Section 7. 3, mean effective green times of such streams are determined as linear functions of the green times and the cycle time. In Section 7. 4, the optimal signal setting with nonoverlapping phases based upon the criterion of minimizing the degree of saturation is dealt with and it is shown that this criterion leads to the rule of thumb that the mean effective green times should be in proportion to the corresponding



ratios of flow to saturation flow. In Section 7. 5, the optimal signal setting with nonoverlapping phases based upon the criterion of minimum overall delay is discussed and an approximation algorithm which is a refinement of Webster's method is derived. In Section 7. 6, the optimal signal settings with overlapping phases based upon the two criteria are dealt with and approximation algorithms are presented.

## 7. 2. Criteria for undersaturation of a whole intersection

Consider an intersection with  $J$  arms which is controlled by a fixed-cycle traffic light. Throughout this chapter, following Webster and Cobbe (1966), a phase means a sequence of conditions applied to one or more streams which, during the cycle, receive simultaneous identical signal indications. On the other hand, a stage means a part of the cycle during which one or more specified streams gain simultaneously right-of-way. Consequently, one green period of a phase may comprise several consecutive stages. Throughout this chapter, suppose that the traffic light have  $N$  phases; note that this loses no generality. If a stream gains right-of-way during the green period of the  $n$ th phase ( $n=1,2,\dots,N$ ), then the stream is said to belong to the  $n$ th phase. Let  $M_n$  streams belong to the  $n$ th phase. The  $m$ th stream belonging to the  $n$ th phase is called the  $(n,m)$  stream ( $n=1,2,\dots,N$ ,  $m=1,2,\dots,M_n$ ) and vehicles which compose it are called briefly  $(n,m)$  vehicles ( $n=1,2,\dots,N$ ,  $m=1,2,\dots,M_n$ ). A stream is called of the first priority in a stage during which it has right-of-way, if it is not obstructed by the other streams including pedestrians which have right-of-way in the same stage. A stream is called of the second priority in a stage during which it has right-of-way, if it is neither of the first priority nor obstructed by other streams which have

right-of-way in the same stage and are not of the first priority. Similarly, a stream of the  $i$ th priority ( $i=3, 4, \dots$ ) can be defined. It is assumed throughout this chapter that vehicles keep to the left side of the road. Thus, in case of the usual four-armed intersections, a stream going straight ahead is of the first priority and a stream turning right is of the second priority. If pedestrians obstruct a stream turning left, the stream is of the second priority. Let some vehicles of the  $i$ th priority ( $i=2, 3, \dots$ ) wait in the queue at the end of the red period. As the light turns green, the head of the queue moves and reaches the position to cross the streams of the higher priorities. This time from the beginning of the green period is called the starting delay for the stream of the  $i$ th priority. If any vehicle of the higher priorities comes into the intersection, the vehicle at the position to cross can not move until that vehicle has passed through. The sum of these blocked time intervals within the green and the amber periods is called the block for the stream. The starting delay and the block are called collectively the lost time for the stream. If no vehicles of the higher priorities come into the intersection, vehicles in the queue cross the lanes of the higher priorities at departure headways and pass through the intersection. Let a long queue remain at the end

of the green period owing to a heavy stream or a long lost time. The time interval between the end of the green period and the instant at which the queue held within the intersection has passed across the lanes of the high priorities is called the clearance period for the stream of the  $i$ th priority. This period is thought of as effectively green. For the stream, therefore, the effective green period is defined as the sum of the green period and the clearance period less the lost time. It should be noted that the starting delay and the clearance period are inherent in the stream and the layout of the intersection but the block is affected by the streams of the higher priorities and varies with the green time, that is, the length of the green period and the cycle time. For the stream of the  $i$ th priority, thus, the effective green time varies with the green time and the cycle time. On the other hand, for a stream of the first priority, the lost time reduces to the starting delay, that is, the acceleration delay for getting the entire queue into motion, because the block vanishes. Moreover, the clearance period corresponds to the time interval between the end of the green period and the instant at which the last vehicle that fails to stop for the amber signal passes the stop line. Hence the effective green period for the stream of the first priority becomes

the sum of the green period and the clearance period less the starting delay. These definitions for the stream of the first priority agree with the conventional ones in substance. Since the conventional definitions are formulated mainly for the streams of the first priority, the definitions adopted in this chapter may be an extension of those conventional ones to the streams of the  $i$ th priority ( $i=2,3,\dots$ ).

Let the cycle time of the traffic light be  $T$ (seconds) and the  $n$ th green time  $G_n$  ( $n=1,2,\dots,N$ ). Suppose that for each stream, lost times and clearance times in all cycles and departure headways of all vehicles are mutually independent and identically distributed random variables with finite means and independent each other. Let  $\ell_{nm}$  (seconds) be the mean lost time,  $c_{nm}$ (seconds) the mean clearance time and  $\mu_{nm}$  the mean departure headway, respectively, for the  $(n,m)$  stream. Then, by definition, the mean effective green time, say,  $g_{nm}$ (seconds) is given by

$$(2.1) \quad g_{nm} = G_n + c_{nm} - \ell_{nm} .$$

Note that  $g_{nm}$  and  $\ell_{nm}$  can be estimated by

$$g_{nm} = \mu_{nm} e_{nm}$$

and

$$\ell_{nm} = G_n + c_{nm} - \mu_{nm} e_{nm} ,$$

where  $e_{nm}$  is the mean number of  $(n,m)$  vehicles which pass through the intersection during one cycle in which the

queue always exists. Furthermore, all streams of arriving vehicles are assumed to be stationary ones without after-effects (see Khintchine (1960)), or generalized Poisson processes (see Parzen (1962), page 127), with finite and positive intensities. Let  $\lambda_{nm}$  (vehicles per second) be the mean arrival number per second of the  $(n,m)$  stream per lane. Under more general assumptions than the above-mentioned ones, it has been shown in Chapter 4 by use of a generalized model of the GI/G/1 queueing process that the queue of the  $(n,m)$  stream has a unique stationary distribution, if and only if

$$(2.2) \quad \lambda_{nm} \mu_{nm} T < G_n + c_{nm} - l_{nm}.$$

That is, this condition is necessary and sufficient for the intersection to be undersaturated with respect to the  $(n,m)$  stream.

Define the degree of saturation of the  $(n,m)$  stream, denoted by  $\rho_{nm}$ , as

$$(2.3) \quad \rho_{nm} = \lambda_{nm} \mu_{nm} T / (G_n + c_{nm} - l_{nm}).$$

Note that this definition agrees completely with that of the "degree of saturation" used popularly. Moreover, the reciprocal of  $\rho_{nm}$  is called the degree of undersaturation of the  $(n,m)$  stream and denoted by  $\gamma_{nm}$ . Put

$$(2.4) \quad y_{nm} = \lambda_{nm} \mu_{nm}.$$

Since  $1/\mu_{nm}$  means the mean saturation flow (vehicles per second) per lane for the  $(n,m)$  stream,  $y_{nm}$  can be thought

of as the ratio of flow to saturation flow for the  $(n,m)$  stream. By (2. 1) and (2. 4), the degree of saturation of the  $(n,m)$  stream can be rewritten as

$$(2. 5) \quad \rho_{nm} = y_{nm} T / \varepsilon_{nm}.$$

Define  $\rho_n (n=1,2,\dots,N)$  by

$$(2. 6) \quad \rho_n = \max_m \{ \rho_{nm} \}$$

and define the degree of saturation of the whole intersection denoted by  $\rho$  as

$$(2. 7) \quad \rho = \max_n \{ \rho_n \}.$$

The stream which attains  $\rho_n$  is called the predominant stream of the  $n$ th phase and the streams which attain  $\rho$  are called the predominant streams.

Clearly, (2. 2) leads directly to the following theorem:

Theorem 1.

The whole intersection is undersaturated if and only if  $\rho < 1$ .

If  $\rho \geq 1$ , then at least one queue grows up indefinitely with time. Moreover from (2. 5), if for some streams  $y_{nm} \geq 1$ , then  $\rho_{nm} \geq 1$ . Therefore it is necessary for undersaturation of the whole intersection that for all streams  $(n,m)$ ,

$$(2. 8) \quad y_{nm} < 1.$$

Define the degree of undersaturation of the whole intersection denoted by  $\gamma$  as

$$(2.9) \quad \gamma = \min_n \{ \gamma_n \},$$

where for  $n=1, 2, \dots, N$ ,

$$(2.10) \quad \gamma_n = \min_m \{ \gamma_{nm} \} = \min_m \{ g_{nm} / y_{nm} T \}.$$

Clearly, the criterion in Theorem 1 is equivalent to

$$(2.11) \quad \gamma > 1.$$

With all vehicle-actuated traffic lights, the maximum length of vehicle-extension period (see, Webster and Cobbe (1966)) of each phase is predetermined in order to prevent vehicles which do not belong to the phase from waiting indefinitely. Therefore, if the traffic is fairly heavy on all phases, the vehicle-actuated traffic light operates in effect as the fixed-cycle one. Thus, in case of the heavy traffic on all phases, Theorem 1 and criterion (2.11) remain valid for the intersection controlled by the vehicle-actuated traffic light.



### 7. 3. Mean effective green time

In the preceding section, the degree of saturation  $\rho_{nm}$  or the degree of undersaturation  $\gamma_{nm}$  is defined in terms of the mean effective green time  $g_{nm}$  and  $g_{nm}$  is given by (2. 1). However, this equation is just the definition and is unsatisfactory particularly for streams of low priorities. Moreover, the analysis of the optimal signal settings discussed in the following sections requires the explicit expression of the mean effective green time or the degree of saturation. Thus, the mean effective green time is dealt with in this section. Denote by  $a_{nm}$  the mean starting delay of the (n,m) stream. Since for the stream of the first priority, the lost time reduces to the starting delay, it follows from (2. 1) that if the (n,m) stream is of the first priority throughout its green period, then

$$(3. 1) \quad g_{nm} = G_n + c_{nm} - a_{nm}$$

Suppose that the nth phase is nonoverlapping; that is, the green period of the nth phase does not overlap with green periods of the other phases. Let the (n,m) stream be of the second priority throughout the green period of the nth phase. Thus, all streams of the first priority that obstruct the (n,m) stream belong to the nth phase and have the green time  $G_n$ . Denote by  $b_{nm}$  the mean

length of the block of the (n,m) stream. Then from (A. 15) in the Appendix II it follows that

$$(3. 2) \quad b_{nm} = (G_n + \hat{c}_{nm}) \{1 - \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) / (1 - \hat{y}_{nm})\} \\ + \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) (\hat{y}_{nm} T + \hat{a}_{nm}) / (1 - \hat{y}_{nm}),$$

where  $\hat{y}_{nm}$ ,  $\hat{a}_{nm}$  and  $\hat{c}_{nm}$  are defined by (A. 9),  $\hat{\lambda}_{nm}$  is given by (A. 11) and  $\mathcal{L}_{nm}$  is the constant time during which the (n,m) vehicle at the head of the queue can not move immediately before any vehicle of the first priority has passed the conflict point with the (n,m) stream. In the above, in order to guarantee the existence of the stationary distributions for the queues of the first priority, it is assumed that

$$(3. 3) \quad \hat{y}_{nm} T < G_n + \hat{c}_{nm} - \hat{a}_{nm}.$$

Moreover, note that the block of the (n,m) stream is measured not at the conflict point where the (n,m) stream is obstructed by the streams of the first priority but at the stop lines of these streams. Since the (n,m) stream conflicts with the streams of the first priority in the center of the intersection in most cases, it is relevant to assume that  $\hat{a}_{nm}$  is nearly equal to the sum of the mean starting delay of vehicles of the first priority,  $\hat{a}_{nm}$  and the mean time it takes them to reach the conflict point from their stop lines. Thus, by (3. 2), the mean lost time of the (n,m) stream,  $\ell_{nm}$  is:

$$\begin{aligned}
(3.4) \quad \ell_{nm} &= a_{nm} - \hat{a}_{nm} + b_{nm} \\
&= (G_n + \hat{c}_{nm}) \{ 1 - \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) / (1 - \hat{y}_{nm}) \} \\
&\quad + \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) (\hat{y}_{nm} T + \hat{a}_{nm}) / (1 - \hat{y}_{nm}) + a_{nm} - \hat{a}_{nm}.
\end{aligned}$$

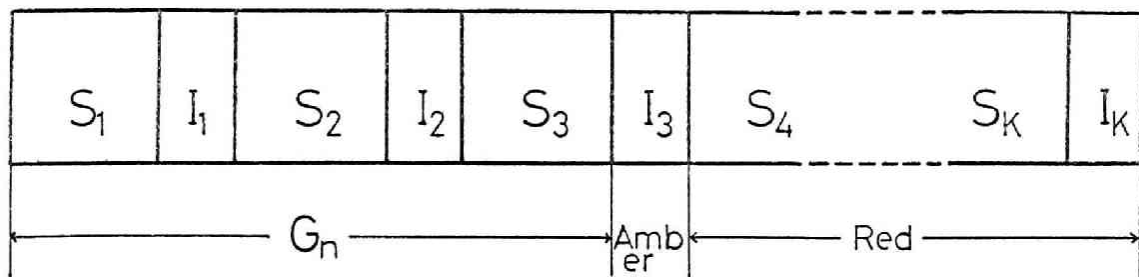
Since it can be assumed with little loss of generality that the mean clearance time of the (n,m) stream is longer than the mean time from the end of the green period until the last vehicle of the first priority reaches the conflict point with the (n,m) stream, that is,  $c_{nm} \geq \hat{c}_{nm} + a_{nm} - \hat{a}_{nm}$ , it follows from (2.1) and (3.4) that for the (n,m) stream which belongs to the nonoverlapping phase,

$$\begin{aligned}
(3.5) \quad g_{nm} &= \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) (G_n - \hat{y}_{nm} T + \hat{c}_{nm} - \hat{a}_{nm}) / (1 - \hat{y}_{nm}) \\
&\quad + (c_{nm} - a_{nm} - \hat{c}_{nm} + \hat{a}_{nm}).
\end{aligned}$$

It is clear that under Condition (3.3),  $g_{nm}$  is positive. Since both the criterion in Theorem 1 and criterion (2.11) can decide whether Condition (3.3) is satisfied or not, (3.5), which is used for estimation of  $\rho$  or  $\gamma$ , remains valid even if Condition (3.3) is not satisfied. Moreover, (3.5) includes (3.1) as a particular case, because if the (n,m) stream is of the first priority, then  $\hat{\lambda}_{nm} = \hat{y}_{nm} = 0$ . Thus, for the (n,m) stream which is of the first or second priority and belongs to the nonoverlapping phase, its mean effective green time,  $g_{nm}$  is given by (3.5). Let us assume that  $\hat{y}_{nm} < 1$  and  $\hat{y}_{nm} T \gg G_j + \hat{c}_{nm} - \hat{a}_{nm}$ . Then the criterion in Theorem 1 leads to  $\rho_{nm} < 0 < 1 \ll \hat{\rho}_{nm} \leq \rho$ ,

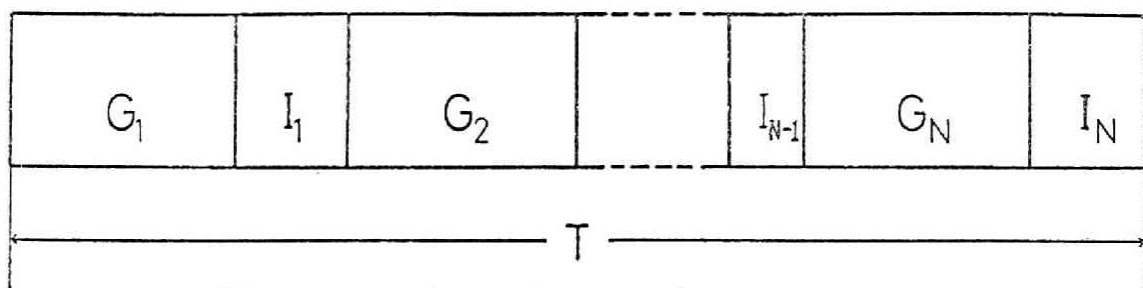
where  $\hat{f}_{nm}$  is the degree of saturation of the stream which attains  $\hat{y}_{nm}$ . Thus, the (n,m) stream which is extremely saturated is deleted from the candidates which attain  $\hat{f}$ . On the other hand, criterion (2. 11) leads to  $\hat{y} \leq \hat{y}_{nm} < 0 < \hat{z}_{nm} \ll 1$ . This shows that criterion (2. 11) is more efficient than the criterion in Theorem 1.

Let the nth phase be overlapping. In this case, the green period of the nth phase may comprise several stages and the priority of the (n,m) stream may vary with stages. Thus, many different circumstances can be thought out. To begin with, consider as a typical and practical example the circumstances in which the green period of the nth phase comprises three stages and the (n,m) stream is of the first priority in the first and the third stages and of the second priority in the second stage (see Fig. 1). Webster and Cobbe (1966) calls the first stage "late release" and the third stage "early cut-off". Denote by  $S_k$  the length of the kth stage ( $k=1,2,3$ ), that is, the kth green period and by  $I_k$  the intergreen period between the kth and the (k+1)st stages. Moreover denote by  $\tilde{t}_{nm}$  the mean time it takes the vehicles of the first priority in the second stage to reach the conflict point with the (n,m) stream from their stop lines, for  $a_{nm}$  means the acceleration delay and can not represent  $\tilde{t}_{nm}$ . It follows from (3. 1), or (3. 5), and (3. 2) that



Overlapping phase

FIG. 1



Nonoverlapping phase

FIG. 2

$$\begin{aligned}
(3.6) \quad g_{nm} &= (S_1 + I_1 + \hat{a}_{nm} + \tilde{t}_{nm} - a_{nm}) + [S_2 + \hat{c}_{nm} - \hat{a}_{nm} \\
&\quad - (S_2 + \hat{c}_{nm}) \{1 - \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) / (1 - \hat{y}_{nm})\} \\
&\quad - \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) (\hat{y}_{nm} T + \hat{a}_{nm}) / (1 - \hat{y}_{nm})] \\
&\quad + (S_3 + c_{nm} + I_2 - \hat{c}_{nm} - \tilde{t}_{nm}) \\
&= G_n + c_{nm} - a_{nm} - [(S_2 + \hat{c}_{nm}) \{1 - \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) / (1 - \hat{y}_{nm})\} \\
&\quad + \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) (\hat{y}_{nm} T + \hat{a}_{nm}) / (1 - \hat{y}_{nm})],
\end{aligned}$$

where  $G_n = \sum_{k=1}^3 S_k + \sum_{k=1}^2 I_k$  and  $c_{nm}$  and  $a_{nm}$  are the mean clearance time and the mean starting delay for the  $(n, m)$  stream of the first priority. In much the same way, the mean effective green time can be determined for all the other situations in which in some stages the  $(n, m)$  stream is of the second priority on. These results are omitted to avoid duplications but they imply that the mean effective green time is a linear function of the lengths of the stages and the cycle time, and is strictly increasing in the lengths of the stages and nonincreasing in the cycle time; that is, if  $G_n = \sum_{k=1}^{k_n} S_{nk} + \sum_{k=1}^{k_n-1} I_{nk}$  and  $I_{nk} (k=1, \dots, k_n-1)$  are constants, then for the  $(n, m)$  stream,

$$\begin{aligned}
(3.7) \quad g_{nm} &\equiv g_{nm}(S_{n1}, S_{n2}, \dots, S_{nk_n}, T), \\
&= \sum_{k=1}^{k_n} u_{nmk} S_{nk} - v_{nm} T + w_{nm},
\end{aligned}$$

where  $u_{nmk} > 0$  and  $v_{nm} \geq 0$ . In particular, for the  $(n, m)$  stream which belongs to the nonoverlapping phase, (3.7)

reduces to

$$(3.8) \quad \begin{aligned} g_{nm} &\equiv g_{nm}(G_n, T) \\ &= u_{nm}G_n - v_{nm}T + w_{nm}, \end{aligned}$$

where  $u_{nm} > 0$  and  $v_{nm} \geq 0$ . From (3.5), for stream (n,m) of the second priority,

$$(3.9) \quad \begin{aligned} u_{nm} &= \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) / (1 - \hat{y}_{nm}), \\ v_{nm} &= \hat{y}_{nm} \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) / (1 - \hat{y}_{nm}) \\ \text{and} \quad w_{nm} &= c_{nm} - a_{nm} + (\hat{c}_{nm} - \hat{a}_{nm}) \{ \exp(-\hat{\lambda}_{nm} \mathcal{L}_{nm}) / (1 - \hat{y}_{nm}) - 1 \}. \end{aligned}$$

Finally, by (3.1), for stream (n,m) which is of the first priority throughout its green period,

$$(3.10) \quad u_{nm} = 1, \quad v_{nm} = 0 \quad \text{and} \quad w_{nm} = c_{nm} - a_{nm}.$$

#### 7. 4. Signal setting to minimize the degree of saturation

Many authors have dealt with the problem of optimal signal settings of fixed-cycle traffic lights. Most of them including Webster (1958), Miller (1963b), Webster and Cobbe (1966) and Allsop (1971 a,b) use minimum overall delay as the criterion in deducing optimal signal settings. In this section, however, a new criterion of minimizing the degree of saturation is adopted. Since the degree of saturation decides, as shown in Theorem 1, whether the intersection is undersaturated or not, this criterion seems suitable for congested intersections. The relation between this criterion and minimum overall delay will be discussed in the following sections. It is assumed in this section that each stream belongs to a fixed phase and all phases are nonoverlapping. Although this assumption restricts signal settings within narrow limits, most traffic lights installed in intersections with 3 or 4 arms satisfy this assumption and hence, many authors have made this assumption in deducing optimal signal settings. Optimal signal settings without this assumption will be dealt with in Section 7. 6.

Let one cycle of the traffic light comprise  $N$  nonoverlapping phases. Denote by  $G_n$  and  $I_n$  ( $n=1,2,\dots,N$ ) the green time of the  $n$ th phase and the intergreen time



between the end of the green period of the  $n$ th phase and the beginning of the green period of the  $(n(\text{mod}.N)+1)$ st phase, . . . respectively (see Fig. 2). Therefore,

$$(4.1) \quad \sum_{n=1}^N G_n = T - I,$$

where  $I = \sum_{n=1}^N I_n > 0$  and for all  $n$ ,

$$(4.2) \quad G_n \geq 0.$$

Moreover it is required that for  $\hat{T} (> I)$ ,

$$(4.3) \quad I \leq T \leq \hat{T},$$

where  $\hat{T}$  is the maximum value of admissible cycle times. Suppose that for all streams,

$$(4.4) \quad 0 < y_{nm} < 1.$$

This assumption loses no generality in view of (2.8). It follows from (2.5) and (3.8) that the degree of saturation of the  $(n,m)$  stream,  $f_{nm}(G_n, T)$  is given by

$$(4.5) \quad f_{nm}(G_n, T) = y_{nm} T / (u_{nm} G_n - v_{nm} T + w_{nm}).$$

Hence, by (2.7), the problem to be solved is :

$$(4.6) \quad \min_{T, \underline{G}} \max_{n, m} \{ f_{nm}(G_n, T) \}$$

subject to (4.1) through (4.3),

where the  $N$  dimensional column vector  $\underline{G} = (G_1, G_2, \dots, G_N)$ .

Denote by  $\rho^*$  the minimum value, if it exists, of this problem.

As noted in the preceding section, however, the degree of saturation is less efficient than the degree of undersaturation. Moreover, minimizing the degree of saturation is equivalent to maximizing the degree of undersaturation. In this sense, it is convenient to rewrite Problem (4. 6) in terms of the degree of undersaturation. It follows from (2. 10) and (3. 8) that the degree of undersaturation of the  $n$ th phase,  $\gamma_n(G_n, T)$  is given by

$$(4. 7) \quad \gamma_n(G_n, T) = \min_m \{g_{nm}(G_n, T)/y_{nm}T\}.$$

Since  $g_{nm}(G_n, T)$  are strictly increasing, continuous and concave in  $G_n$ ,  $\gamma_n(G_n, T)$  is also strictly increasing, continuous and concave in  $G_n$  (see, Mangasarian (1969), page 61). Thus, Problem (4. 6) is rewritten as:

$$(4. 8) \quad \max_{T, G} \min_n \{\gamma_n(G_n, T)\}$$

subject to (4. 1) through (4. 3).

Denote by  $\gamma^*$  and  $(G^*, T^*)$  the maximum value and the optimal solutions, if they exist, of this problem. If  $\gamma^* > 0$ , then  $\rho^* = 1/\gamma^*$  and  $(G^*, T^*)$  is the optimal solutions of Problem (4. 6). If  $\gamma^* \leq 0$ , then  $\rho^* > 1$ ,  $\rho^* \neq 1/\gamma^*$  and  $(G^*, T^*)$  is not the optimal solution of Problem (4. 6): from Theorem 1 and (2. 11), however, both  $\rho^* > 1$  and  $\gamma^* \leq 0$  imply

that the intersection is oversaturated with any admissible signal setting. Note that when the intersection is known to be oversaturated with all admissible signal settings, how much the intersection is oversaturated is of little importance. Clearly, Problem (4. 8) is equivalent to:

$$(4. 9) \quad \max_T \{ \gamma^*(T) \}$$

subject to (4. 3),

where

$$(4. 10) \quad \gamma^*(T) = \max_{\underline{G}} \min_n \{ \gamma_n(G_n, T) \}$$

subject to (4. 1) and (4. 2).

Denote by  $\underline{G}^*(T)$  the optimal solution, if exists, of Inner Maximization Problem (4. 10).

Now consider Inner Maximization Problem(4.10) for appropriately fixed  $T$  satisfying (4.3). Before discussing this problem, consider the following problem which is slightly more general than the above:

$$(4. 11) \quad \max_{\underline{X}} \min_n \{ f_n(x_n) \}$$

subject to

$$(4. 12) \quad \sum_{n=1}^N x_n = a$$

$$(4. 13) \quad \text{and } \underline{x} = (x_1, x_2, \dots, x_N) \in X,$$

where  $X$  is a subset of  $N$  dimensional Euclidean space and  $f_n(x_n)$  ( $n=1,2,\dots, N$ ) is strictly increasing in  $x_n$ . Then it can be shown that the following lemma holds for this generalized problem:

Lemma

Suppose that there exists a point  $\hat{x}=(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N)$  which satisfies (4. 12), (4. 13) and

$$(4. 14) \quad f_1(\hat{x}_1)=f_2(\hat{x}_2)=\dots=f_N(\hat{x}_N).$$

Then this point  $\hat{x}$  is the unique optimal solution of Problem (4. 11) through (4. 13).

Proof.

Assume that there exists another point  $\bar{x}=(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N) \neq \hat{x}$  which satisfies (4. 12) through (4. 14). Since there exists some  $n$  such that  $\hat{x}_n > \bar{x}_n$  ( $\hat{x}_n < \bar{x}_n$ ) and all  $f_n$  are strictly increasing functions, it follows from (4. 14) that for all  $n$ ,  $f_n(\hat{x}_n) > f_n(\bar{x}_n)$  ( $f_n(\hat{x}_n) < f_n(\bar{x}_n)$ ).

Consequently, for all  $n$ ,  $\hat{x}_n > \bar{x}_n$  ( $\hat{x}_n < \bar{x}_n$ ). By (4. 12), this implies that  $a > a$  ( $a < a$ ). Therefore  $\hat{x}$  is unique.

Suppose that another point  $\tilde{x}=(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N) \neq \hat{x}$  is a solution of Problem (4. 11) through (4. 13); that is, for some  $\ell$  and all  $n \neq \ell$ ,

$$f_\ell(\tilde{x}_\ell) > f_n(\tilde{x}_n) \geq f_\ell(\hat{x}_\ell) = f_n(\hat{x}_n),$$

where the first inequality is due to the uniqueness of

$\hat{x}$ . Consequently  $\tilde{x}_\ell > \hat{x}_\ell$  and  $\tilde{x}_n \geq \hat{x}_n$ . By (4. 12), this implies that  $a > a$ . Hence  $\hat{x}$  is the unique solution of Problem (4. 11) through (4. 13). The proof is concluded.

Let  $[n]$  denote the predominant stream of the  $n$ th phase; that is,

$$(4. 15) \quad \gamma_n(G_n, T) = g_n(G_n, T)/y_n T.$$

Then by (3. 8),

$$(4. 16) \quad g_n(G_n, T) = u_n G_n - v_n T + w_n,$$

where  $u_n > 0$  and  $v_n \geq 0$ . Here, it is to be noted that the predominant stream  $[n]$  may vary with  $G_n$  and  $T$ . From Lemma , in order to solve Problem (4. 10), it suffices to show that a point  $\hat{G} = (\hat{G}_1, \hat{G}_2, \dots, \hat{G}_N)$ , if it exists, is determined by (4. 1) and

$$(4. 17) \quad \gamma_1(\hat{G}_1, T) = \gamma_2(\hat{G}_2, T) = \dots = \gamma_N(\hat{G}_N, T)$$

satisfies (4. 2). By (4. 15), these equations can be rewritten as

$$(4. 18) \quad g_1(\hat{G}_1, T)/y_1 = g_2(\hat{G}_2, T)/y_2 = \dots = g_N(\hat{G}_N, T)/y_N.$$

Thus, by (4. 16), for arbitrarily fixed  $n$  and all  $\ell \neq n$ ,

$$(4. 19) \quad \hat{G}_\ell = \{y_\ell u_n \hat{G}_n + (y_n v_\ell - y_\ell v_n)T + y_\ell w_n - y_n w_\ell\} / y_n u_\ell.$$

Substituting these equations into (4. 1) yields for

$n=1,2,\dots,$

$$(4.20) \quad \hat{G}_n = \left\{ \left( 1 - \sum_{n=1}^N v_n/u_n + (v_n/y_n) \sum_{n=1}^N y_n/u_n \right) T - \left( 1 - \sum_{n=1}^N w_n/u_n + (w_n/y_n) \sum_{n=1}^N y_n/u_n \right) \right\} / \left\{ (u_n/y_n) \sum_{n=1}^N y_n/u_n \right\}.$$

Therefore, the point  $\hat{G}$  which satisfies (4.1) and (4.18) exists certainly by the strict increase and the continuity of  $g_n(G_n, T)$  and is uniquely determined as (4.20), although formally. Now suppose that

$$(4.21) \quad \sum_{n=1}^N v_n/u_n < 1$$

and

$$(4.22) \quad T \geq \max_n \left\{ \left[ 1 - \sum_{n=1}^N w_n/u_n + (w_n/y_n) \sum_{n=1}^N y_n/u_n \right] / \left[ 1 - \sum_{n=1}^N v_n/u_n + (v_n/y_n) \sum_{n=1}^N y_n/u_n \right] \right\}.$$

Then these assumptions imply that  $\hat{G}$  satisfies (4.2). Hence it follows from Lemma that  $\hat{G}$  given by (4.20) is the unique solution  $G^*(T)$  of Problem (4.10). Moreover, by (4.1), (4.10) and (4.17),

$$(4.23) \quad \gamma^*(T) = \left\{ 1 - \sum_{n=1}^N v_n/u_n - \left( 1 - \sum_{n=1}^N w_n/u_n \right) / T \right\} / \sum_{n=1}^N y_n/u_n.$$

Note that  $\gamma^*(T)$  is the minimum value of the right-hand sides of (4.23) with  $u_n$ ,  $v_n$  and  $w_n$  replaced by  $u_{nm}$ ,  $v_{nm}$  and  $w_{nm}$  of all streams belonging to the  $n$ th phase.

### Theorem 2.

Under Conditions (4. 21) and (4. 22), Problem (4. 10) has the unique optimal solution given by (4. 20) and the maximum value given by (4. 23).

It should be noted that the optimal solution  $\underline{g}^*(T)$  of Problem (4. 10) has been determined by (4. 1) and (4.18); (4.18) means that the mean effective green times  $g_n$  should be in proportion to the corresponding ratios of flow to saturation flow  $y_n$ . This is just the well-known rule of thumb (see, for example, Highway capacity manual (1950), page 92, or Evans (1950), page 224). Moreover, Problem (4. 10) and Lemma show that (4. 13) is derived from only the criterion of maximizing the degree of undersaturation, as long as mean effective green times are strictly increasing in green times. Note that continuity of  $g_n$  leads to existence of  $\hat{\underline{g}}$  and Conditions (4. 21) and (4. 22) lead to feasibility of  $\hat{\underline{g}}$ . These facts explain that the rule of thumb mentioned above is derived from the criterion of maximizing the degree of undersaturation or, equally, that of minimizing the degree of saturation.

### Corollary 1.

For a fixed cycle time, the criterion of minimizing the degree of saturation leads to the rule that the mean effective green times should be in proportion to the

corresponding ratios of flow to saturation flow.

Let us assume that Conditions (4. 21) and (4. 22) are satisfied at  $T=\hat{T}$ . This means that there exists a set of  $T$  including  $\hat{T}$ , say,  $S_T$  on which (4. 3), (4. 21) and (4. 22) are satisfied. Suppose that for all  $T \in S_T$ ,

$$(4. 24) \quad I > \sum_{n=1}^N w_n/u_n.$$

Since for all  $n$ ,  $\gamma_n(G_n, T)$  is continuous in both  $G_n$  and  $T$  and for  $T \in S_T$ ,  $\gamma^*(T)$  is determined by (4. 1) and (4. 17),  $\gamma^*(T)$  is continuous in  $T \in S_T$ . Moreover, (4. 23) and (4. 24) imply  $\gamma^*(T)$  is strictly increasing in  $T \in S_T$ .

It is clear that for  $T \notin S_T$ ,  $\gamma^*(T)$  is less than the right member of (4. 23). Consequently from (4. 9) and Theorem 2 it follows that Problem (4. 8) has the unique optimal solution  $T^*=\hat{T}$  and  $G^*=G^*(\hat{T})$ , which is given by (4. 20) with  $T$  replaced by  $\hat{T}$ . Clearly, by (4. 23),

$$(4. 25) \quad \gamma^* = \left\{ 1 - \sum_{n=1}^N v_n/u_n - (I - \sum_{n=1}^N w_n/u_n)/\hat{T} \right\} / \sum_{n=1}^N y_n/u_n.$$

### Theorem 3.

Suppose that Conditions (4. 21) and (4. 22) are satisfied at  $T=\hat{T}$ . Under Condition (4. 24), Problem (4. 8) attains the maximum value given by (4. 25) at  $T=T^*$  and  $G_n=G_n^*$  ( $n=1, 2, \dots, N$ ), where

$$(4. 26) \quad T^* = \hat{T}$$



and

$$(4.27) \quad G_n^* = \left\{ \left( 1 - \sum_{n=1}^N v_n / u_n + (v_n / y_n) \sum_{n=1}^N y_n / u_n \right) \hat{T} - \left( I - \sum_{n=1}^N w_n / u_n + (w_n / y_n) \sum_{n=1}^N y_n / u_n \right) \right\} / \left\{ (u_n / y_n) \sum_{n=1}^N y_n / u_n \right\}.$$

It follows from (4.25) that

$$\lim_{T \rightarrow \infty} \eta^* = (1 - \sum_{n=1}^N v_n / u_n) / \sum_{n=1}^N y_n / u_n.$$

Since (4.24) implies  $\eta^*$  is strictly increasing in  $\hat{T}$  and  $\eta^*$  is not larger than the right hand side of (4.25) with  $u_n$ ,  $v_n$  and  $w_n$  replaced by  $u_{nm}$ ,  $v_{nm}$  and  $w_{nm}$  of any stream belonging to the  $n$ th phase, Theorem 3 leads to:

#### Corollary 2.

Suppose that (4.24) is satisfied for all  $T > I$ . If

$$\sum_{n=1}^N (v_n + y_n) / u_n \geq 1 \text{ for sufficiently large } T,$$

then the intersection is oversaturated with any possible signal setting.

Let us assume that the total intergreen time  $I > 0$  is predetermined in such a way that for all  $T \in S_T$ ,

$$(4.28) \quad I \geq \sum_{n=1}^N w_n / u_n + [\max_n (v_n / y_n) (1 - \sum_{n=1}^N (w_n + y_n) / u_n)]^+,$$

where  $[\cdot]^+ = \max\{0, \cdot\}$ . Clearly, (4.28) implies (4.24).

Thus, Corollary 2 suggests the following condition should be made: for all  $T \in S_T$ ,

$$(4. 29) \quad \sum_{n=1}^N (v_n + y_n)/u_n < 1.$$

By (4. 4), this condition is much stronger than (4. 21).  
Therefore from (4. 22) and Theorem 3 it follows that if

$$(4. 30) \quad \hat{T} \geq T_0 \equiv \max_H \left\{ \left[ 1 - \sum_{n=1}^N w_n/u_n + (w_n/y_n) \sum_{n=1}^N y_n/u_n \right] / \right. \\ \left. \left[ 1 - \sum_{n=1}^N v_n/u_n + (v_n/y_n) \sum_{n=1}^N y_n/u_n \right] \right\},$$

then  $(Q^*, T^*)$  given by (4. 26) and (4. 27) is the unique solution of Problem (4. 8). Put for  $T = \hat{T}$ ,

$$(4. 31) \quad T_1 = (1 - \sum_{n=1}^N w_n/u_n) / (1 - \sum_{n=1}^N (v_n + y_n)/u_n).$$

Then (4. 4), (4. 16) and (4. 28) imply  $T_1 \geq T_0$ . Thus, by (4. 25), if  $\hat{T} > T_1$ , then  $\eta^* > 1$ ; criterion (2. 8) shows that the intersection is undersaturated with the optimal signal setting  $(Q^*, T^*)$ . Moreover, if  $T_0 \leq \hat{T} \leq T_1$ , then  $\eta^* \leq 1$  and the intersection is saturated with respect to at least one stream belonging to each phase: that is, the intersection is uncontrollable under constraint (4. 3), as long as no streams are allowed to queue up indefinitely. Put for  $T = \hat{T}$ ,

$$(4. 32) \quad T_2 = (1 - \sum_{n=1}^N w_n/u_n) / (1 - \sum_{n=1}^N v_n/u_n).$$

Then, by (4. 25),  $\hat{T} > \max(T_0, T_2)$  implies  $\eta^* > 0$ . As noted in the remark following (4. 8), if  $\eta^* > 0$ , then  $\rho^* = 1/\eta^*$  and  $(Q^*, T^*)$  is the unique optimal solution of Problem (4. 6).

Therefore, if  $\hat{T} > \max(T_0, T_2)$ , then

$$(4.33) \quad \rho^* = \hat{T} \sum_{n=1}^N (y_n / u_n) / \left\{ (1 - \sum_{n=1}^N v_n / u_n) \hat{T} - (I - \sum_{n=1}^N w_n / u_n) \right\}.$$

Note that  $T_1 > T_2$  and that  $T_0$ ,  $T_1$  and  $T_2$  may be smaller than  $I$ .

#### Theorem 4.

Suppose that Conditions (4.28) and (4.29) are satisfied.

If  $\hat{T} > T_1$ , then Problem (4.6) attains the minimum value

$\rho^* < 1$  at the unique optimal solution  $(\underline{G}^*, T^*)$  and the

intersection is undersaturated with the optimal signal

setting  $(\underline{G}^*, T^*)$ ; If  $\max(T_0, T_2) < \hat{T} \leq T_1$ , then Problem (4.6)

attains the minimum value  $\rho^* \geq 1$  at the unique optimal

solution  $(\underline{G}^*, T^*)$  and the intersection is oversaturated

with the optimal signal setting  $(\underline{G}^*, T^*)$ ; If  $\hat{T} \leq \max(T_0, T_2)$ ,

then the intersection is extremely oversaturated with all

admissible signal settings; where  $\rho^*$  is given by (4.33),

$(\underline{G}^*, T^*)$  is given by (4.26) and (4.27),  $T_0$ ,  $T_1$  and  $T_2$

are defined by (4.30) through (4.32).

Let predominant streams of all phases be of the first priority. This case is fundamental and almost all authors

have restricted their discussion within this case. By

(3.10), for stream  $(n, m)$  of the first priority,

$$(4.34) \quad \zeta_{nm}(G_n, T) = (G_n / y_{nm} + w_{nm} / y_{nm}) / T.$$

Since the differences of  $w_{nm}$  are negligible for all streams

of the first priority which belong to the  $n$ th phase, put

$$w_{nm}=w_n, \text{ for all } (n,m) \text{ of the first priority.}$$

Then since  $\gamma_{nm}(-w_n, T)=0$ , the predominant stream of the  $n$ th phase is determined as:

$$(4.35) \quad \text{for } G_n \geq [-w_n]^+ \text{ and all } T > I, \\ y_n = \max\{y_{nm}\}$$

$$\text{and} \quad \text{for } 0 \leq G_n < [-w_n]^+ \text{ and all } T > I, \\ (4.36) \quad y_n = \min\{y_{nm}\},$$

where max or min is taken over all streams of the first priority which belong to the  $n$ th phase. Suppose that

$$(4.37) \quad \hat{T} \geq L \equiv I - \sum_{n=1}^N w_n.$$

Note that  $L=T_2$ , because of (3.10) and (4.32). Therefore  $\gamma^* \geq 0$  and  $g_n(G_n^*, T^*) \geq 0$  implies  $G_n^* \geq -w_n$ . Thus, the predominant stream  $[n]$  at  $T=T^*$ , which is used in (4.21) through (4.33), is given by (4.35) under (4.37). Then Conditions (4.28) and (4.29) reduce to:

$$(4.38) \quad Y \equiv \sum_{n=1}^N y_n < 1$$

and

$$(4.39) \quad I \geq \sum_{n=1}^N w_n + \max_n (w_n/y_n)(1-Y).$$

Moreover, by (4.30) and (4.31),

$$(4.40) \quad T_0 = L + Y \max_n (w_n/y_n)$$

and

$$(4.41) \quad T_1 = L/(1-Y).$$

Hence it follows from Theorem 4 that under Conditions (4.38) and (4.39), if  $\hat{T} > T_1$ , then the unique optimal solution of Problem (4.6) is  $(\underline{G}^*, T^*)$  and

$$(4.42) \quad \rho^* = Y\hat{T}/(\hat{T}-L) < 1,$$

where  $T_1$  is given by (4.40), and  $T^* = \hat{T}$  and for  $n=1, 2, \dots, N$ ,

$$(4.43) \quad G_n^* = y_n(\hat{T}-L)/(Y-w_n).$$

It is to be noted that  $\rho^* > Y$  for all  $T^*$ . Hence if  $\hat{T}$  is not predetermined and for given value  $\rho_0$  such that  $1 > \rho_0 > Y$ , it is required to set  $\rho^* = \rho_0$ , then by (4.42),  $T^*$  should be determined as

$$(4.44) \quad T^* = L\rho_0/(\rho_0 - Y).$$

Consider the case in which streams of the second priority or higher priorities are included in the predominant stream  $[n]$  which varies with  $G_n$  and  $T$ . Although it is possible to deal with this general case in a similar manner to the case discussed above, it seems to be rather complicated, because which stream to adopt as  $[n]$  at  $T=T^*$  and  $\underline{G}=\underline{G}^*$  must be determined. Thus, let us discuss directly Problem (4.8) in what follows, because this problem is more efficient than Problem (4.6). To begin with, (4.7) and (4.8) imply that Problem (4.8) is equivalent to:

$$\begin{aligned}
(4.45) \quad & \max_{\tau, \xi, \eta} \{ \eta \} \\
& \text{subject to (4.1) through (4.3) and for all} \\
& n=1, 2, \dots, N \text{ and all } m=1, 2, \dots, M_n, \\
& y_{nm} T \eta \leq \xi_{nm}(G_n, T).
\end{aligned}$$

Consequently it follows from (3.8) that Problem (4.8) is equivalent to the following nonlinear programming problem:

$$\begin{aligned}
(4.46) \quad & \max_{\tau, \xi, \eta} \{ \eta \} \\
& \text{subject to} \\
& y_{nm} \eta T + v_{nm} T - u_{nm} G_n \leq w_{nm} \quad (n=1, 2, \dots, N, \quad m=1, 2, \dots, M_n), \\
& T - \sum_{n=1}^N G_n = I, \\
& I \leq T \leq \hat{T} \\
& \text{and } G_n \geq 0 \quad (n=1, 2, \dots, N).
\end{aligned}$$

Put  $\tau = 1/T$  and  $\theta_n = G_n/T$  ( $n=1, 2, \dots, N$ ). Then this nonlinear programming problem can be rewritten as:

$$\begin{aligned}
(4.47) \quad & \min_{\tau, \xi, \eta} \{ \eta \} \\
& \text{subject to} \\
& -y_{nm} \eta + u_{nm} \theta_n + w_{nm} \tau \geq v_{nm} \quad (n=1, 2, \dots, N, \quad m=1, 2, \dots, M_n), \\
& \sum_{n=1}^N \theta_n + I\tau = 1, \\
& 1/\hat{T} \leq \tau \leq 1/I \\
& \text{and } \theta_n \geq 0 \quad (n=1, 2, \dots, N).
\end{aligned}$$

Clearly, this problem can be solved by linear programming (see, for example, Dantzig (1963) or Orchard-Hays (1968)).

Suppose that Conditions (4. 21), (4. 24) and (4. 30) are satisfied for all predominant streams which vary with  $\underline{G}$  at  $T=\hat{T}$ . Then from Theorem 3 it follows that Problem (4. 47) reduces to the following simple linear programming problem:

$$\begin{aligned}
 (4. 48) \quad & \max_{\underline{\theta}, \hat{\tau}} \{ \hat{\tau} \} \\
 & \text{subject to} \\
 & y_{nm} \hat{\tau} - u_{nm} \theta_n \leq w_{nm} \hat{\tau} - v_{nm} \quad (n=1, 2, \dots, N, \quad m=1, 2, \dots, M_n), \\
 & \sum_{n=1}^N \theta_n = 1 - I \hat{\tau} \\
 & \text{and } \theta_n \geq 0 \quad (n=1, 2, \dots, N),
 \end{aligned}$$

where  $\hat{\tau} = 1/\hat{T}$ . Theorem 3 shows that this problem has the unique optimal solution. Let  $\pi_0$  and  $\pi_{nm}$  ( $n=1, 2, \dots, N$ ,  $m=1, 2, \dots, M_n$ ) be the dual variables corresponding to the equality constraint and the inequality constraints of (4. 48). The dual problem of (4. 48) (see, for example, Dantzig (1963), page 126) is:

$$\begin{aligned}
 (4. 49) \quad & \min_{\pi_0, \pi_n} \left\{ \pi_0 (1 - I \hat{\tau}) + \sum_{n=1}^N \pi_n (\hat{\tau} w_n - v_n) \right\} \\
 & \text{subject to} \\
 & \sum_{n=1}^N \pi_n x_n = 1, \\
 & \pi_n u_n \leq \pi_0 \quad (n=1, 2, \dots, N) \\
 & \text{and } \pi_n \geq 0 \quad (n=1, 2, \dots, N),
 \end{aligned}$$

where row vectors  $\pi_n = (\pi_{n1}, \pi_{n2}, \dots, \pi_{nM_n})$  and column vectors  $x_n = (x_{n1}, x_{n2}, \dots, x_{nM_n})$ ,  $u_n = (u_{n1}, u_{n2}, \dots, u_{nM_n})$ ,  $v_n = (v_{n1}, v_{n2}, \dots, v_{nM_n})$  and  $w_n = (w_{n1}, w_{n2}, \dots, w_{nM_n})$ . Since Conditions (4. 21), (4. 24) and (4. 30) appear to be satisfied for practical cases, it suffices to solve Linear Programming Problem (4. 48) or (4. 49) in order to solve Problem (4. 6) or (4. 8). Finally it should be noted that all results in this section are valid with little modification for Problem (4. 6) with constraint  $T = \hat{T}$  instead of (4. 3). In particular, Theorems 3 and 4, Corollaries 1 and 2 Equations (4. 33), (4. 42) and (4. 43) and Problems (4. 48) and (4. 49) are valid without modification.



#### 7. 5. Webster's optimal signal setting.

In the preceding section, the optimal signal setting based upon the criterion of minimizing the degree of saturation has been discussed on the probable assumption that each stream belongs to a fixed phase and all phases are nonoverlapping. On the same assumption, the optimal signal setting based upon the criterion of minimum overall delay is dealt with in this section. Webster (1958) discussed this type of optimal signal settings, although he dealt with only the predominant streams of the first priority, and developed the method of optimal signal setting, which is used popularly due to its simplicity and compatibility. The derivation of his method, however, seems to be slightly rough. The purpose of this section is to analyze his method in some detail. Throughout this section, the same notations as in the preceding section are used.

In order to proceed with the optimal signal setting mentioned above, it is essential to obtain the mean delay to all vehicles arriving at the intersection in unit time. Several theoretical results on the mean delay per vehicle at intersections controlled by fixed-cycle traffic lights have been obtained by Newell (1960, 1965), Miller (1963), Buckley and Wheeler (1964), Darroch (1964) and McNeil (1968). Their expressions for the mean

delay per vehicle, however, appear too complicated to deduce the optimal signal setting, except Miller's: his expression requires measurement of the variance of arrivals. On the other hand, Webster (1958) derived a simple empirical formula from computer simulation and tested his formula by comparison with observed data. Thus, in what follows, Webster's delay formula is used:

$$(5.1) \quad d_{nm} = \frac{9}{10} \left\{ T(1 - g_{nm}/T)^2 / 2(1 - g_{nm} \rho_{nm}/T) + \rho_{nm}^2 / 2 \lambda_{nm}(1 - \rho_{nm}) \right\},$$

where  $d_{nm}$  is the mean delay per  $(n,m)$  vehicle. This formula can be rewritten in terms of  $\rho_{nm}$  as follows:

$$(5.2) \quad d_{nm} = \frac{9}{20} \left\{ T(1 - y_{nm}/\rho_{nm})^2 / (1 - y_{nm}) + \rho_{nm}^2 / \lambda_{nm}(1 - \rho_{nm}) \right\}.$$

Note that this formula is valid only if  $0 < \rho_{nm} < 1$ ; this condition implies  $y_{nm} < 1$ . Denote by  $D(\underline{G}, T)$  the overall mean delay of vehicles arriving in unit time when the signal setting is  $(\underline{G}, T)$ . Since  $\lambda_{nm} d_{nm}$  is the total mean delay of  $(n,m)$  vehicles arriving in unit time,

$$(5.3) \quad D(\underline{G}, T) = \frac{9}{20} \sum_{n=1}^N D_n(\underline{G}_n, T),$$

where for  $n=1, 2, \dots, N$ ,

$$(5.4) \quad D_n(G_n, T) = \sum_{m=1}^{M_n} \left\{ \lambda_{nm} T (1 - y_{nm} / \rho_{nm}(G_n, T))^2 / (1 - y_{nm}) + \rho_{nm}^2(G_n, T) / (1 - \rho_{nm}(G_n, T)) \right\}$$

and for all  $m$ ,  $\rho_{nm}(G_n, T)$  is defined by (4.5) and satisfies

$$(5.5) \quad 0 < \rho_{nm}(G_n, T) < 1.$$

Suppose that constraint (4.3) on  $T$  is imposed. Since (4.1) and (4.2) must be satisfied, the problem to be solved is:

$$(5.6) \quad \min_{T, G} \{ D(G, T) \}$$

subject to (4.1), (4.2), (4.3) and (5.5).

To begin with, it must be decided whether this problem has feasible solutions or not, that is, its consistency must be decided. This question can be definitely answered by solving Problem (4.6) or (4.8) or (4.47). That is, if  $0 < \rho^* < 1$ , then Problem (5.6) has feasible solutions, otherwise it has no feasible solutions. The more precise result follows from Theorem 4.

#### ~~Theorem 5~~

Suppose that Conditions (4.28) and (4.29) are satisfied. Then Problem (5.6) has feasible solutions, if and only if for predominant streams at  $T = \hat{T}$ ,

$$(5.7) \quad \hat{T} > T_1 \equiv (I - \sum_{n=1}^N w_n / u_n) / (1 - \sum_{n=1}^N (v_n + y_n) / u_n).$$

In the sequel of this section, suppose that Conditions (4. 28), (4. 29) and (5. 7) are satisfied. Clearly, (4. 28) and (4. 29) imply that for all  $n$  and  $m$ ,

$$(5. 8) \quad \rho_{nm}(G_n, T) \geq y_{nm}.$$

Moreover,  $\rho^*(\leq 1)$  can be obtained by solving Linear Programming Problem (4. 48) or (4. 49) which is considerably simpler than (4. 47). Since  $D(\underline{G}, T)$  is continuous in both  $\underline{G}$  and  $T$  and strictly increasing with respect to  $\rho_{nm}$  for fixed  $T$ , Problem (5. 6) has a minimum value. Denote by  $\bar{D}$  and  $(\bar{\underline{G}}, \bar{T})$  the minimum value and the optimal solutions of Problem (5. 6).

It is clear that Problem (5. 6) is equivalent to:

$$(5. 9) \quad \min_{\bar{T}} \{ \bar{D}(T) \}$$

subject to (4. 3),

where

$$(5. 10) \quad \bar{D}(T) = \min_{\bar{\underline{G}}} \left\{ \frac{1}{20} \sum_{n=1}^N D_n(G_n, T) \right\}$$

subject to (4. 1), (4. 2) and (5. 5).

In order to decide whether Inner Minimization Problem (5. 10) for fixed  $T$  has feasible solutions or not, similarly to Problem (5. 6), it suffices to solve:

$$(5. 11) \quad \rho_{(T)}^* = \min_{\bar{\underline{G}}} \max_n \{ \rho_n(G_n, T) \}$$

subject to (4. 1) and (4. 2),

where  $\rho_n(G_n, T)$  is defined by (2. 6).

Note that this problem reduces to Linear Programming Problem (4. 48) or (4. 49) with  $\hat{\tau}$  replaced by  $1/T$ .

Define  $\hat{T}_1$  as

$$(5. 12) \quad \hat{T}_1 = \inf\{T; \rho^*(T) < 1, 1 \leq T \leq \hat{T}\}.$$

Since  $\rho^*(T) = 1/\eta^*(T)$  is continuous and strictly decreasing in  $T$  due to (4. 28), (5. 7) implies that  $\hat{T}_1$  exists certainly. Note that  $\hat{T}_1$  is not smaller than  $T_1$  given by (5. 7), because  $\hat{T}_1$  is the maximum of  $I$  and the last member of (5. 7) defined for predominant streams at  $T = \hat{T}_1$ . For  $T$  such that  $\hat{T}_1 < T \leq \hat{T}$ , define  $X(T)$  as the set composed of points  $\underline{G}$  such that (4. 1), (4. 2) and (5. 5) are satisfied. Since (4. 28) and (4. 29) imply that (4. 2) is a non-binding constraint, the set  $X(T)$  is open and convex. It follows from (5. 4) that

$$(5. 13) \quad \partial D_n(G_n, T) / \partial G_n = - \sum_m \left\{ 2\lambda_{nm} u_{nm} (\rho_{nm}(G_n, T) - y_{nm}) / \right. \\ \left. (1 - y_{nm}) \rho_{nm}(G_n, T) + u_{nm} \rho_{nm}^3(G_n, T) (2 - \rho_{nm}(G_n, T)) / \right. \\ \left. y_{nm} T (1 - \rho_{nm}(G_n, T))^2 \right\}$$

and

$$(5. 14) \quad \partial^2 D_n(G_n, T) / \partial G_n^2 = \sum_m \left\{ 2\lambda_{nm} u_{nm}^2 / (1 - y_{nm}) T \right. \\ \left. + 2u_{nm}^2 \rho_{nm}^4(G_n, T) (\rho_{nm}^2(G_n, T) - 3\rho_{nm}(G_n, T) + 3) / \right. \\ \left. y_{nm}^2 T^2 (1 - \rho_{nm}(G_n, T))^3 \right\}.$$

Since by (5. 14),  $D(\underline{G}, T)$  is strictly convex in  $\underline{G} \in X(T)$  and

for  $\underline{G}$  on the boundary set of  $X(T)$ ,  $D(\underline{G}, T) = \infty$ , Inner Minimization Problem (5. 10) for  $T$  such that  $\hat{T}_1 < T \leq \hat{T}$  has a unique optimal solution, which is denoted by  $\bar{G}(T)$ . Although Inner Minimization Problem (5. 10) can be solved by dynamic programming (see, for example, Bellman and Dreyfus (1962), Mine and Ohno(1970 a,b) or separable programming (see, for example, Dantzig (1963), pp. 482-490), these methods consume much time. Thus it may be practically important to develop a method which supplies a good approximate solution of Problem (5. 6), i.e. (5. 9) and (5. 10).

Webster (1958) considered a four-armed intersection with two-phases and calculated  $D(\underline{G}, T)$  with only two predominant streams for various values of cycle times and ratios of effective green times in order to find the properties of  $\bar{G}(T)$ . His conclusion is that optimal solutions of Inner Minimization Problem (5. 10) are rather close to approximate solutions obtained by the rule of thumb that the mean effective green times should be in proportion to the corresponding ratios of flow to saturation flow. He derived the approximate equation of  $\bar{D}(T)$  from substituting those approximate solutions for  $\bar{G}(T)$  and, by differentiating the equation with respect to  $T$ , obtained the approximate solution of  $(\bar{G}, \bar{T})$ . Thus, Corollary 1 implies that Webster's method may be regarded as the approximate method of the following strategy:

For  $T$  such that  $\hat{T}_1 < T \leq \hat{T}$ , solve Problem (5. 11), that is, Linear Programming Problem (4. 48) or (4. 49) with  $\hat{\tau}$  replaced by  $1/T$ , instead of Inner Minimization Problem (5. 10), to obtain the approximate solution  $\underline{G}^*(T)$  for  $\underline{G}(T)$ . Then solve One-dimensional Minimization Problem (5. 9) with  $\bar{D}(T)$  estimated by  $D(\underline{G}^*(T), T)$ . This approximation strategy is more accurate and more useful than Webster's method particularly in case several streams including ones of the second priority belong to each phase. It has not been shown, however, under what conditions this strategy or Webster's method supplies a good approximation to the optimal solution of Problem (5. 6). That is, it is open to question under what conditions  $\underline{G}^*(T)$  becomes a good approximate value of the optimal solution  $\underline{G}(T)$ . In order to discuss this problem, put

$$(5. 15) \quad \Delta \underline{G}(T) = \underline{G}(T) - \underline{G}^*(T).$$

Clearly,

$$(5. 16) \quad \sum_{n=1}^N \Delta G_n(T) = 0$$

and by the classical Lagrange multiplier method (see, for example, Hadley (1964), pp. 60f.),  $\Delta \underline{G}(T)$  satisfies for all  $n \neq \ell$

$$(5. 17) \quad \partial D_n(\underline{G}^*(T) + \Delta \underline{G}_n(T), T) / \partial G_n = \partial D_\ell(\underline{G}^*(T) + \Delta \underline{G}_\ell(T), T) / \partial G_\ell$$

Define  $\tilde{G}_n(T)$  ( $n=1, 2, \dots, N$ ) by  $f_n(\tilde{G}_n(T), T)=1$  for the  $n$ th predominant stream at  $(G_n^*(T), T)$ . Since  $G_n^*(T)$  and  $f^*(T)=f_n(G_n^*(T), T)$  ( $n=1, 2, \dots, N$ ) are given by (4. 27) and (4. 33), respectively, with  $\hat{T}$  replaced by  $T$ , (4. 5) yields

$$G_n^*(T) - \tilde{G}_n(T) = y_n T (1 - f^*(T)) / u_n f^*(T).$$

Thus

$$(5. 13) \quad \Delta G_n(T) > -y_n T (1 - f^*(T)) / u_n f^*(T),$$

since  $\tilde{G}_n(T)$  must satisfy  $f_n(\tilde{G}_n(T), T) < 1$  and  $f_n(G_n, T)$  is strictly decreasing in  $G_n$ . Therefore, (5. 16) and (5. 18) imply

$$(5. 19) \quad \max_n \{|\Delta G_n(T)|\} < \left( \sum_{n=1}^N y_n / u_n \right) T (1 - f^*(T)) / f^*(T).$$

Suppose that for an appropriately small positive number  $\delta_1$ ,

$$(5. 20) \quad 1 - f^*(T) \leq \delta_1.$$

Then from (4. 16), (4. 33) and (5. 19),

$$(5. 21) \quad \max_n \{|\Delta G_n(T)|\} < \left( 1 - \sum_{n=1}^N v_n / u_n \right) T \delta_1.$$

Hence, under Condition (5. 20), (5. 17) can be approximated as follows : for all  $n \neq l$ ,

$$(5. 22) \quad \begin{aligned} & \partial D_n^* / \partial G_n + (\partial^2 D_n^* / \partial G_n^2) \Delta G_n(T) \\ & = \partial D_l^* / \partial G_l + (\partial^2 D_l^* / \partial G_l^2) \Delta G_l(T), \end{aligned}$$



where  $\partial D_n^*/\partial G_n$  and  $\partial^2 D_n^*/\partial G_n^2$  denote  $\partial D_n(G_n^*(T), T)/\partial G_n$  and  $\partial^2 D_n(G_n^*(T), T)/\partial G_n^2$ . Solving (5. 16) and (5. 22) yields for  $n=1, 2, \dots, N$ ,

$$(5. 23) \quad \Delta G_n(T) = \left\{ \sum_{\ell=1}^N (\partial D_\ell^*/\partial G_\ell - \partial D_n^*/\partial G_n) / \partial^2 D_\ell^*/\partial G_\ell^2 \right\} / \left( \partial^2 D_n^*/\partial G_n^2 \right) \left\{ \sum_{\ell=1}^N 1 / (\partial^2 D_\ell^*/\partial G_\ell^2) \right\}.$$

Since  $\partial^2 D_n^*/\partial G_n^2 > 0$  for all  $n$ , (5. 23) leads to

$$(5. 24) \quad \sum_{n=1}^N |\Delta G_n(T)| = \left\{ \sum_{n=1}^N \left| \sum_{\ell=1}^N (\partial D_\ell^*/\partial G_\ell - \partial D_n^*/\partial G_n) / \partial^2 D_\ell^*/\partial G_\ell^2 \right| / (\partial^2 D_n^*/\partial G_n^2) \right\} / \left\{ \sum_{n=1}^N 1 / (\partial^2 D_n^*/\partial G_n^2) \right\}.$$

A feasible solution  $\underline{G}$  is said to be an  $\varepsilon$ -near-optimal solution of Inner Minimization Problem (5. 10), if for positive number  $\varepsilon$ ,

$$(5. 25) \quad \sum_{n=1}^N |\bar{G}_n(T) - G_n| \leq \varepsilon.$$

Therefore, since from (5. 24),

$$(5. 26) \quad \sum_{n=1}^N |\Delta G_n(T)| \leq \max_{n, \ell} |\partial D_n^*/\partial G_n - \partial D_\ell^*/\partial G_\ell| \sum_{n=1}^N 1 / (\partial^2 D_n^*/\partial G_n^2)$$

and by (5. 14),

$$(5. 27) \quad 1 / (\partial^2 D_n^*/\partial G_n^2) < (y_n^2 T^2 / 2u_n^2) (1 - \rho^*(T))^3 / (\rho^*(T))^4,$$

Condition (5. 20) implies that  $\underline{G}^*(T)$  is an  $\varepsilon$ -near-optimal solution, if

$$(5.28) \quad \max_{n, \ell} |\partial D_n^* / \partial G_n - \partial D_\ell^* / \partial G_\ell| \left( \sum_{n=1}^N y_n^2 / 2u_n^2 \right) T^2 \delta_1^3 \leq \varepsilon (1 - \delta_1)^4.$$

Since from (5.13) and (5.20),

$$(5.29) \quad \max_{n, \ell} |\partial D_n^* / \partial G_n - \partial D_\ell^* / \partial G_\ell| \\ < \max_{n, \ell} \left| u_n / y_n - u_\ell / y_\ell + \sum_{m \neq [n]} u_{nm} (\rho_{nm}^*)^3 (2 - \rho_{nm}^*) (1 - \rho^*)^2 / \right. \\ \left. y_{nm} (1 - \rho_{nm}^*)^2 (\rho^*)^3 (2 - \rho^*) - \sum_{k \neq [\ell]} u_{\ell k} (\rho_{\ell k}^*)^3 (2 - \rho_{\ell k}^*) (1 - \rho^*)^2 / \right. \\ \left. y_{\ell k} (1 - \rho_{\ell k}^*)^2 (\rho^*)^3 (2 - \rho^*) \right| / T \delta_1^2,$$

where  $\rho^*$  and  $\rho_{nm}^*$  denote  $\rho^*(T)$  and  $\rho_{nm}^*(G_n^*(T), T)$ , it follows from (5.28) that  $\underline{G}^*(T)$  is an  $\varepsilon$ -near-optimal solution of Inner Minimisation Problem (5.10), if  $\delta_1$  is sufficiently small, that is, the intersection is nearly oversaturated.

Now a feasible solution  $\underline{G}$  is said to be an  $\varepsilon$ -optimal solution of Inner Minimization Problem (5.10), if for  $\varepsilon > 0$ ,

$$(5.30) \quad |\bar{D}(T) - D(\underline{G}, T)| \leq \varepsilon.$$

Since  $D(\underline{G}, T)$  is strictly convex in  $\underline{G} \in X(T)$ ,

$$|\bar{D}(T) - D(\underline{G}^*(T), T)| < \left| \sum_{n=1}^N (\partial D_n^* / \partial G_n) \Delta G_n(T) \right|,$$

It follows from (5.23) that

$$(5.31) \quad |\bar{D}(T) - D(\underline{G}^*(T), T)| < \frac{9}{20} \max_{n, \ell} |\partial D_n^* / \partial G_n - \partial D_\ell^* / \partial G_\ell| \cdot \\ \sum_{n=1}^N |\partial D_n^* / \partial G_n| / (\partial^2 D_n^* / \partial G_n^2).$$

Thus,  $\underline{G}^*(T)$  is an  $\varepsilon$ -optimal solution under Condition (5.20), if

$$(5.32) \quad \max_{n,l} \left| \partial D_n^* / \partial G_n - \partial D_l^* / \partial G_l \right| \sum_{n=1}^N \left\{ (y_n / u_n)^2 \sum_{m=1}^{M_n} u_{nm} / y_{nm} \right\} T \delta_1 \\ \leq 4\varepsilon(1-\delta_1)^4,$$

because due to (5.13) and (5.14),

$$(5.33) \quad \left| \partial D_n^* / \partial G_n \right| / \left( \partial^2 D_n^* / \partial G_n^2 \right) \leq (y_n / u_n)^2 \left( \sum_{m=1}^{M_n} u_{nm} / y_{nm} \right) T \cdot \\ (1 - \rho^*(T)) / 2.$$

In the above, Condition (5.20) has been assumed. However, even if (5.20) is not satisfied,  $\Delta G(T)$  given by (5.23) will remain valid, because, in fact,  $\max_n \{\Delta G_n(T)\}$  is expected to be not so large, as shown also by examples of Webster (1958) and Allsop (1971a). Thus, (5.24), (5.26) and (5.31) will remain valid without Condition (5.20). Now consider the practical case that all streams (n,m) satisfy

$$(5.34) \quad y_{nm} + 0.05 \leq \rho_{nm}^* \leq 0.7.$$

For this case, the direct calculation of (5.13) and (5.14) yields

$$(5.35) \quad \partial D_n^* / \partial G_n = - \sum_{m=1}^{M_n} 2\lambda_{nm} u_{nm} (1 - y_{nm} / \rho_{nm}^*) / (1 - y_{nm}) + O(1/T)$$

and

$$(5.36) \quad \partial^2 D_n^* / \partial G_n^2 = \sum_{m=1}^{M_n} 2\lambda_{nm} u_{nm}^2 / (1 - y_{nm}) T + O(1/T^2),$$

since  $T$  is considerably larger than the other parameters. Therefore, from (5.26), (5.31), (5.35) and (5.36) it

follows that in case of (5. 34),

$$(5. 37) \quad \sum_{n=1}^N |\Delta G_n(T)| \leq \max_{n, \ell} \left| \sum_m \lambda_{nm} u_{nm} (1-y_{nm}/\rho_{nm}^*) / (1-y_{nm}) \right. \\ \left. - \sum_k \lambda_{\ell k} u_{\ell k} (1-y_{\ell k}/\rho_{\ell k}^*) / (1-y_{\ell k}) \right| \cdot \\ \sum_{n=1}^N [1 / \{ \sum_m \lambda_{nm} u_{nm}^2 / (1-y_{nm}) \}]$$

and

$$(5. 38) \quad |\bar{D}(T) - D(\tilde{G}^*(T), T)| \\ < T \max_{n, \ell} \left| \sum_m \lambda_{nm} u_{nm} (1-y_{nm}/\rho_{nm}^*) / (1-y_{nm}) \right. \\ \left. - \sum_k \lambda_{\ell k} u_{\ell k} (1-y_{\ell k}/\rho_{\ell k}^*) / (1-y_{\ell k}) \right| \cdot \\ \sum_{n=1}^N [ \{ \sum_m \lambda_{nm} u_{nm} (1-y_{nm}/\rho_{nm}^*) / (1-y_{nm}) \} / \{ \sum_m \\ \lambda_{nm} u_{nm}^2 / (1-y_{nm}) \} ].$$

Suppose that for all streams (n,m) and  $\delta_2$  such that  $0 \leq \delta_2 \leq [0.65 - \max_{n,m} y_{nm}]^+$ ,

$$(5. 39) \quad 0.05 \leq \rho_{nm}^* - y_{nm} \leq 0.05 + \delta_2.$$

Combination of (5. 38) and (5. 39) leads to

$$(5. 40) \quad |\bar{D}(T) - D(\tilde{G}^*(T), T)| < T (0.05 + \delta_2)^2 \max_n \left\{ \sum_m \lambda_{nm} u_{nm} / \right. \\ \left. (1-y_{nm})(y_{nm} + 0.05) \right\} \sum_{n=1}^N [ \{ \sum_m \lambda_{nm} u_{nm} / (1-y_{nm})(y_{nm} + 0.05) \} / \\ \{ \sum_m \lambda_{nm} u_{nm}^2 / (1-y_{nm}) \} ].$$

This means  $\underline{G}^*(T)$  is an  $\varepsilon$ -optimal solution of Inner Minimization Problem (5.10), if  $\delta_2$  is appropriately small, that is, the intersection is moderately undersaturated. In particular, if the number of streams belonging to each phase is equal and if any stream belonging to each phase has the same parameters  $y, \lambda, u, v$  and  $w$  as the corresponding stream belonging to another phase, then by Theorem 4 and (5.13),  $\max_{n, \ell} |\partial D_n^* / \partial G_n - \partial D_\ell^* / \partial G_\ell| = 0$ . Due to (5.16) and (5.17), this implies that  $\underline{G}^*(T)$  is the optimal solution of Inner Minimization Problem (5.10).

The above discussion has justified the approximation strategy or Webster's method to some extent. Thus, let us deal with this strategy in more detail. Since Problem (5.11) reduces to Linear Programming Problem (4.48) or (4.49) with  $\hat{\tau}$  replaced by  $1/T$ ,  $\rho^*(T)$  and  $\underline{G}^*(T)$  for all  $T$  such that  $\hat{T}_1 < T \leq \hat{T}$  can be obtained by parametric linear programming. Moreover, the interval of cycle time during which the predominant streams at  $\underline{G}^*(T)$  do not change can be easily determined by parametric linear programming, because the predominant streams at  $\underline{G}^*(T)$  correspond to the basic variables of Dual Problem (4.49) with  $\hat{\tau}$  replaced by  $1/T$ . Denote by  $M(T)$  the minimum value of cycle times at which the predominant streams at  $\underline{G}^*(T)$  are preserved. It is clear that for some  $T'$  such that  $\hat{T}_1 < T' \leq \hat{T}$ ,  $\underline{G}^*(T)$  is twice continuously

differentiable at  $T \in (M(T'), T')$ . Consequently,  $D(\underline{Q}^*(T), T)$  is also twice continuously differentiable at  $T \in (M(T'), T')$  and

$$(5.41) \quad dD(\underline{Q}^*(T), T)/dT = \sum_{n=1}^N \sum_{m=1}^{M_n} \left\{ \lambda_{nm} (1 - y_{nm} / \rho_{nm}^*) (1 - y_{nm} / \rho_{nm}^* - 2A_{nm}/T) / (1 - y_{nm}) - A_{nm} (\rho_{nm}^*)^3 (2 - \rho_{nm}^*) / y_{nm} T^2 (1 - \rho_{nm}^*)^2 \right\},$$

where  $A_{nm} = y_{nm} u_{nm} (I - \sum_{n=1}^N w_n / u_n) / (u_n \sum_{n=1}^N y_n / u_n) + (u_{nm} w_n - u_n w_{nm}) / u_n$ .

Moreover, the direct calculation of  $d^2 D(\underline{Q}^*(T), T)/dT^2$  shows that  $D(\underline{Q}^*(T), T)$  is strictly convex in  $T \in (M(T'), T')$ . Thus, the approximation strategy leads to the following algorithm:

Step 1. Set  $k=1$  and  $T^1 = \bar{T} = \hat{T}$ .

Step 2. Solve Problem (5.11), i.e. Linear Programming Problem (4.48) or (4.49) with  $\hat{\tau}$  replaced by  $1/T^k$  and obtain  $M(T^k)$  and for  $T \in [M(T^k), T^k]$ ,  $\rho^*(T)$  and  $\underline{Q}^*(T)$ .

Step 3. Set  $T^{k+1} = \max\{\hat{T}_1, M(T^k)\}$ , where  $\hat{T}_1$  is defined by (5.12).

Step 4. Set  $T'$  as what is indicated in the following, according as three exclusive cases: If  $dD(\underline{Q}^*(T), T)/dT|_{T=T^k-0} \leq 0$ , then  $T' = T^k$ ; If  $dD(\underline{Q}^*(T), T)/dT|_{T=T^{k+1}+0} \geq 0$ , then  $T' = T^{k+1}$ ; If  $dD(\underline{Q}^*(T), T)/dT|_{T=T^{k+1}+0} < 0 < dD(\underline{Q}^*(T), T)/dT|_{T=T^k-0}$ , then  $T' = \{T; dD(\underline{Q}^*(T), T)/dT|_{T=T^k-0} = dD(\underline{Q}^*(T), T)/dT|_{T=T^{k+1}+0}\}$ .

$dT=0, T^{k+1} < T < T^k$ , where  $dD(\underline{G}^*(T), T)/dT$  is given by (5. 41).

Step 5. Reset  $\bar{T}$  as what attains  $\min\{D(\underline{G}^*(\bar{T}), \bar{T}), D(\underline{G}^*(T'), T')\}$ .

Step 6. If  $T^{k+1} = \hat{T}_1$ , then  $(\underline{G}^*(\bar{T}), \bar{T})$  is an approximate solution of (5. 6). Otherwise, set  $k$  as  $k+1$  and return to Step 2.

It is to be noted that if the right-hand term in (5. 26) (or (5. 31)) for  $(\underline{G}^*(T'), T')$  in Step 4 is less than  $\varepsilon$  in all iterations, then  $(\underline{G}^*(\bar{T}), \bar{T})$  is an  $\varepsilon$ -near-optimal (or  $\varepsilon$ -optimal) solution of Problem (5. 6). Moreover, if these right-hand terms are not small, then  $(\underline{G}^*(\bar{T}) + \Delta \underline{G}(\bar{T}), \bar{T})$  will give a more accurate value to the solution of Problem (5. 6) than  $(\underline{G}^*(\bar{T}), \bar{T})$ , where  $\Delta \underline{G}(\bar{T})$  is given by (5. 23).

Let us take only those predominant streams of all phases into account that are of the first priority. This is the case discussed by Webster (1958). In this case,  $D_n(G_n, T)$  defined by (5. 4) reduces to

$$(5. 42) \quad D_n(G_n, T) = \lambda_n T (1 - y_n / \rho_n(G_n, T))^2 / (1 - y_n) \\ + \rho_n^2(G_n, T) / (1 - \rho_n(G_n, T)),$$

where predominant stream  $[n]$  is invariant with  $(G_n, T)$  such that  $G_n \geq [-w_n]^+$  and  $T \geq I$ , and is given by (4. 35). Moreover, under Conditions (4. 38) and (4. 39),  $\rho^*(T)$  and  $\underline{G}^*(T)$  are given by (4. 42) and (4. 43), respectively, with  $\hat{T}$  replaced by  $T$ , for  $T > T_1$ , where  $T_1$  is given by (4. 41). Since

$\rho^*(T) \geq 1$  for  $T$  such that  $1 \leq T \leq T_1$ ,  $\hat{T}_1$  defined by (5. 12) is identical with  $T_1$ . Consequently, it follows from (4. 42), (5. 3) and (5. 42) that  $D(\underline{G}^*(T), T)$  is twice continuously differentiable in  $T$  such that  $\hat{T}_1 = T_1 < T \leq \hat{T}$  and

$$(5. 43) \quad dD(\underline{G}^*(T), T)/dT = \frac{9}{20} \left[ \sum_{n=1}^N \{ \lambda_n (YT - y_n T + y_n L) (YT - y_n T - y_n L) / (1 - y_n) Y^2 T^2 \} - NY^2 LT (2T - YT - 2L) / (T - L)^2 (T - YT - L)^2 \right].$$

Since as noted in the above,  $D(\underline{G}^*(T), T)$  is strictly convex in  $T$  such that  $\hat{T}_1 < T \leq \hat{T}$ , if a root of  $dD(\underline{G}^*(T), T)/dT = 0$  exists between  $\hat{T}_1$  and  $\hat{T}$ , then the root is just the solution of the approximation algorithm mentioned above. Since it is difficult to solve (5. 43), Webster (1958) adopted an approximate solution of (5. 43) as his optimal setting of cycle time. His optimal cycle time, however, seems to be shorter for light traffic and considerably longer for heavy traffic than the optimal cycle time given by the present approximation algorithm, which is the exact solution of (5. 43).



## 7. 6. Optimal signal settings: Overlapping phases

In this section, optimal signal settings are discussed in which each phase is permitted to be overlapping. To begin with, it is assumed that each phase is composed of one or more specified stages and each stream, which belongs to a specified phase, has a given priority in each stage of the phase. Let one cycle of the traffic light be composed of  $K$  stages. Denote by  $S_k$  ( $k=1, 2, \dots, K$ ) the green time of the  $k$ th stage and by  $I_k$  the intergreen period between the  $k$ th and the  $(k(\text{mod}.1)+1)$ st stages, as mentioned in Section 7.3. Therefore, for  $K$  dimensional column vector  $\underline{S}=(S_1, S_2, \dots, S_K)$ , the signal setting to be determined is completely represented by  $(\underline{S}, T)$  (see, Fig. 1). Clearly,

$$(6. 1) \quad \sum_{k=1}^K S_k = T - I,$$

where  $I = \sum_{k=1}^K I_k$  and for all  $k$ ,

$$(6. 2) \quad S_k \geq 0.$$

In order to simplify the suffixes used in this section, let us number the  $(n, m')$  stream as  $m = \sum_{i=1}^{n-1} M_i + m'$  and call it the  $m$ th stream. Thus the suffix  $nm$  used in the preceding sections reduces to  $m$ . Put  $M = \sum_{n=1}^N M_n$  and define  $p_{mk}$  ( $m=1, 2, \dots, M, k=1, 2, \dots, K$ ) as one if the  $m$ th stream has right-of-way in the  $k$ th stage and as zero, otherwise. Then, for all  $n$  and  $m$  such that  $\sum_{i=1}^{n-1} M_i < m \leq \sum_{i=1}^n M_i$ ,

$$G_n = \sum_{k=1}^K p_{mk} S_k + \sum I_k,$$

where the last sum is over  $k$  such that  $p_{mk}=1$  and  $p_{mk+1}=1$ .

From (3. 7) it follows that the effective green time of the  $m$ th stream at  $(\underline{S}, T)$ ,  $g_m(\underline{S}, T)$  can be written as

$$(6. 3) \quad g_m(\underline{S}, T) = \sum_{k=1}^K u_{mk} p_{mk} S_k - v_m T + w_m,$$

where  $u_{mk} > 0$  and  $v_m \geq 0$ .

Consequently, by (2. 5),

$$(6. 4) \quad f_m(\underline{S}, T) = y_m T / \left( \sum_{k=1}^K u_{mk} p_{mk} S_k - v_m T + w_m \right).$$

It can be assumed similarly to (4. 4) that for all  $m$ ,

$$(6. 5) \quad 0 < y_m < 1.$$

Suppose that constraint (4. 3) on  $T$  is imposed. Then, in much the same way as in Section 4, the optimal signal setting based upon the criterion of minimizing the degree of saturation can be obtained by solving the following problem:

$$(6. 6) \quad \max_{T, \underline{S}} \left\{ \min_m \left( \sum_{k=1}^K u_{mk} p_{mk} S_k - v_m T + w_m \right) / y_m T \right\}$$

subject to (4. 3), (6. 1) and (6. 2).

Denote by  $\eta^*$  and  $(\underline{S}^*, T^*)$  the maximum value and the optimal solutions of this problem. The minimum value of the degree of saturation,  $\rho^*$  is given by  $\rho^* = 1/\eta^*$  except for abnormal cases that  $\eta^* \leq 0$ . Put  $\tau = 1/T$  and  $\theta_k = S_k/T$  ( $k=1, 2,$

...,K). Then, (6. 6) reduces to the following linear programming problem:

$$\begin{aligned}
 (6. 7) \quad & \max_{\tau, \theta, \eta} \{ \eta \} \\
 & \text{subject to} \\
 & w_m \tau - y_m \eta + \sum_{k=1}^K u_{mk} p_{mk} \theta_k \geq v_m \quad (m=1, 2, \dots, M), \\
 & I \tau + \sum_{k=1}^K \theta_k = 1 \\
 & 1/\hat{T} \leq \tau \leq 1/I \\
 & \text{and } \theta_k \geq 0 \quad (k=1, 2, \dots, K).
 \end{aligned}$$

Since this problem has feasible solutions,  $(\underline{S}^*, T)$  and  $\mathcal{P}^*$  can be obtained by linear programming. Now suppose that

$$(6. 8) \quad I > K \max_m [w_m]^+ / \min_m \left\{ \sum_{k=1}^K u_{mk} p_{mk} \right\}.$$

Denote by  $\eta^*(T) \equiv 1/\mathcal{P}^*(T)$  and  $\underline{S}^*(T)$  the maximum value and the optimal solution of the inner maximization problem defined similarly to (4. 10). That is, for  $T$  satisfying (4. 3),

$$(6. 9) \quad \eta^*(T) = \min_m \left\{ \sum_{k=1}^K u_{mk} p_{mk} S_k^*(T) / y_m T + w_m / y_m T - v_m / y_m \right\},$$

where  $\underline{S}^*(T)$  satisfies (6. 1) and (6. 2). Since

$S_k^1 = (T + \Delta T) S_k^*(T) / T + \Delta T I / K T$  ( $k=1, 2, \dots, K$ ) satisfy (6. 1) with  $T$  replaced by  $(T + \Delta T)$  and (6. 2), from (6. 3) and (6. 9) it follows that for  $\Delta T > 0$ ,

$$\eta^*(T + \Delta T) \geq \min_m \left\{ \sum_{k=1}^K u_{mk} p_{mk} S_k^1 / y_m (T + \Delta T) + w_m / y_m (T + \Delta T) - v_m / y_m \right\}$$

$$z \gamma^*(T) + \{\Delta T / KT(T + \Delta T)\} \min_m \left\{ \left( \sum_{k=1}^K u_{mk} p_{mk} \right) I - Kw_m \right\} / y_m > \gamma^*(T).$$

Thus, under (6. 8),  $\gamma^*(T)$  is strictly increasing in  $T$ .

Hence, under (6. 8), Problem (6. 7) reduces to:

$$(6. 10) \quad \max_{\underline{S}, \gamma} \{ \gamma \}$$

subject to

$$y_m \gamma - \sum_{k=1}^K u_{mk} p_{mk} \theta_k \leq w_m \hat{\tau} - v_m \quad (m=1, 2, \dots, M)$$

$$\sum_{k=1}^K \theta_k = 1 - I \hat{\tau}$$

$$\text{and } \theta_k \geq 0 \quad (k=1, 2, \dots, K),$$

where  $\hat{\tau} = 1/\hat{T}$ . In a similar way to (4. 49), the dual problem of (6. 10) is:

$$(6. 11) \quad \min_{\pi_0, \pi_m} \left\{ \pi_0 (1 - I \hat{\tau}) + \sum_{m=1}^M \pi_m (w_m \hat{\tau} - v_m) \right\}$$

subject to

$$\sum_{m=1}^M \pi_m y_m = 1,$$

$$\sum_{m=1}^M \pi_m u_{mk} p_{mk} \leq \pi_0 \quad (k=1, 2, \dots, K)$$

$$\text{and } \pi_m \geq 0 \quad (m=1, 2, \dots, M),$$

where  $\pi_0$  is the dual variable corresponding to the equality constraint and  $\pi_m$  ( $m=1, 2, \dots, M$ ) is the dual variable corresponding to the  $m$ th inequality constraint. Therefore, if (6. 8) is satisfied, then it suffices to solve Linear

Programming Problem (6. 10) or (6. 11) in order to solve Problem (6. 6) or (6. 7).

Let us deal with the optimal signal setting based on the criterion of minimal overall delay. Put for

$\rho_m(\underline{S}, T)$  given by (6. 4),

$$(6. 12) \quad D_m(\underline{S}, T) = \lambda_m T (1 - y_m / \rho_m(\underline{S}, T))^2 / (1 - y_m) + \rho_m^2(\underline{S}, T) / (1 - \rho_m(\underline{S}, T))$$

and

$$(6. 13) \quad D(\underline{S}, T) = \frac{9}{20} \sum_{m=1}^M D_m(\underline{S}, T).$$

Similarly to (5. 5),  $\rho_m(\underline{S}, T)$  must satisfy

$$(6. 14) \quad 0 < \rho_m(\underline{S}, T) < 1.$$

Assuming constraint (4. 3) on  $T$ , the problem to be solved is:

$$(6. 15) \quad \min_{T, \underline{S}} \{D(\underline{S}, T)\}$$

subject to (4. 3), (6. 1), (6. 2) and (6. 14).

This problem can be solved by the method of feasible directions (Zoutendijk (1960)) or the gradient projection method (Rosen (1960)) or sequential unconstrained minimization techniques (Fiacco and McCormick (1968)). Allsop (1971a) discussed this problem and developed a method for solving it. His method is very close to the method of feasible directions. All of the above methods may take much

time to solve Problem (6. 15). Thus it may be worth developing a method which supplies a good approximation of the optimal solutions of Problem (6. 15). The results in Section 5 suggest that the approximation strategy in the section will supply a good approximation of Problem (6. 15). Suppose that Condition (6. 8) is satisfied. The approximation strategy leads to the following algorithm:

Step 1. Solve Linear Programming Problem (6. 10) or (6. 11).

If  $\rho^* \geq 1$  or  $\leq 0$ , then Problem (6. 15) has no feasible solutions. Otherwise, set  $k=1$  and  $T^1 = \bar{T} = \hat{T}$  and go to Step 2.

Step 2. Solve (6. 10) or (6. 11) with  $\hat{\tau}$  replaced by  $1/T^k$  by parametric linear programming and obtain  $M(T^k)$  defined in Section 5 and for  $T \in [M(T^k), T^k]$ ,  $\rho^*(T)$  and  $\underline{S}^*(T)$ .

Step 3. If  $0 < \rho^*(M(T^k)) < 1$  and  $M(T^k) \geq 1$ , then set  $T^{k+1} = M(T^k)$ . Otherwise, set  $T^{k+1} = \hat{T}_1$ , where  $\hat{T}_1$  is defined by (5. 12).

Step 4. Solve

$$\min_T \{D(\underline{S}^*(T), T)\}$$

Subject to  $T^{k+1} \leq T \leq T^k$

and set  $T'$  as its optimal solution.

Step 5. Reset  $\bar{T}$  as what attains  $\min \{D(\underline{S}^*(\bar{T}), \bar{T}), D(\underline{S}^*(T'), T')\}$ .

Step 6. If  $T^{k+1} = \hat{T}_1$ , then  $(\underline{S}^*(\bar{T}), \bar{T})$  is an approximate

solution of (6. 15). Otherwise, set  $k$  as  $k+1$  and return to Step 2.

Note that the one dimensional minimization problem in Step 4 can be solved by direct search or Rosenbrock's method (1960) or Powell's method (1964). Although Condition (6. 8) is assumed in the above algorithm, the approximation algorithm without (6. 8) is very similar to the above. This approximation algorithm may be useful, since it is difficult to apply Webster's method to signal settings with overlapping phases.

In the above optimal signal settings have been dealt with in which each stream belongs to a fixed phase composed of one or more specified stages. Let us briefly discuss more general optimal signal settings, in which only priority is fixed for each stream. That is, if the  $m$ th stream is predetermined to be of the first priority, then it is not obstructed by the other streams in its green period: if the  $m$ th stream is predetermined to be the  $i$ th ( $i=2,3,\dots$ ) priority, then it is not obstructed by another stream except ones of the higher priorities in its green period. Note that if in a part of the green period of the  $m$ th stream of the  $i$ th priority, no streams of the higher priorities have right-of-way, then the  $m$ th stream becomes in effect of the first priority in the part of its green period. Let one cycle of the traffic light

begin at the beginning of the green period of the first stream. Denote by  $G_m$  ( $m=1,2,\dots,M$ ) the length of the green period of the  $m$ th stream and by  $x_m$  ( $m=1,2,\dots,M$ ) the time from the beginning of the cycle to the beginning of the  $m$ th green period. Let  $A_m$  ( $m=1,2,\dots,M$ ) be the fixed positive length of the amber period of the  $m$ th stream, which follows its green period. Then the signal setting is completely represented by  $(\underline{G}, \underline{x}, T)$  for  $M$  dimensional column vectors  $\underline{G}=(G_1, G_2, \dots, G_M)$  and  $\underline{x}=(x_1, x_2, \dots, x_M)$  (see, Fig. 3). Clearly,

Fig.3

$$(6.16) \quad x_1=0 \text{ and } 0 \leq x_m < T \quad (m=2,3,\dots,M).$$

Moreover,

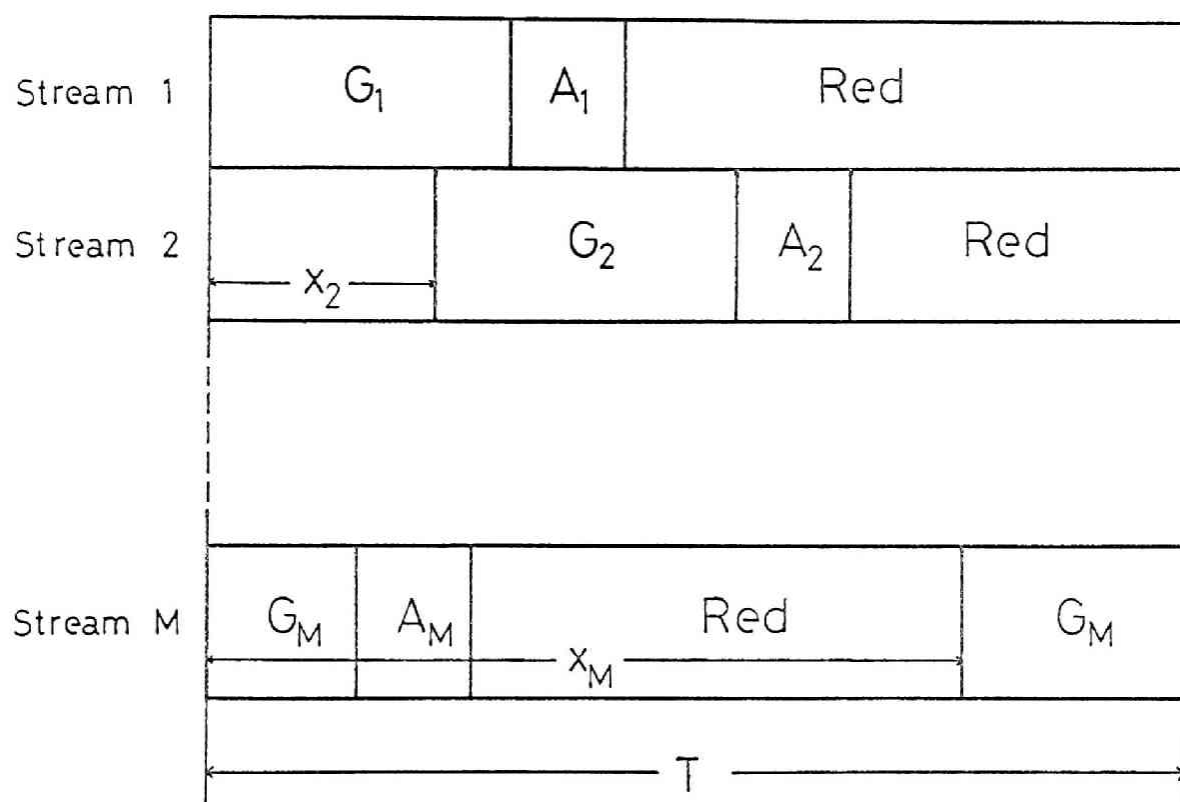
$$(6.17) \quad 0 \leq G_m \leq T - A_m \quad (m=1,2,\dots,M).$$

It follows from the assumption on priority that if the trajectories of the  $m$ th and the  $\ell$ th streams of the same priority intersect, then their green periods must not be overlapping. Therefore, for all such  $m$  and  $\ell$  and sufficiently large  $R$ ,

$$(6.18) \quad \begin{aligned} x_m - x_\ell + G_m + A_m &\leq R z_{m\ell}, \\ x_\ell + G_\ell + A_\ell - x_m - T &\leq R z_{m\ell}, \\ x_\ell - x_m + G_\ell + A_\ell &\leq R(1 - z_{m\ell}), \\ x_m + G_m + A_m - x_\ell - T &\leq R(1 - z_{m\ell}) \end{aligned}$$

and  $z_{m\ell} = 0 \text{ or } 1.$





General overlapping phase

FIG. 3

These inhibition constraints may be imposed for streams of different priorities in view of safety. If the  $m$ th stream is of the first priority, then from (3. 8) and (3. 10), its mean effective green time  $g_m(\underline{G}, \underline{x}, T)$  at signal setting  $(\underline{G}, \underline{x}, T)$  is given by

$$(6. 19) \quad g_m(\underline{G}, \underline{x}, T) = G_m + w_m.$$

For the  $m$ th stream of the second priority on,  $g_m(\underline{G}, \underline{x}, T)$  is a combination of the form of (3. 1) and that of (3. 5). Suppose that for some positive constant  $I$ , constraint (4. 3) on  $T$  is imposed. Then the optimal signal setting based upon the criterion of minimizing the degree of saturation can be obtained by solving the following problem:

$$(6. 20) \quad \begin{aligned} & \max_{\underline{G}, \underline{x}, T, \gamma} \{ \gamma \} \\ & \text{subject to (4. 3), (6. 16) through (6. 18) and} \\ & \quad g_m(\underline{G}, \underline{x}, T) \geq \gamma_m T \gamma \quad (m=1, 2, \dots, M). \end{aligned}$$

Hence, if only streams of the first priority are taken into account, then by (6. 19), Problem (6. 20) reduces to the following mixed integer programming problem:

$$(6. 21) \quad \begin{aligned} & \max_{\theta, \hat{T}, \tau, \gamma} \{ \gamma \} \\ & \text{subject to} \\ & \quad 1/\hat{T} \leq \tau \leq 1/I, \\ & \quad 0 \leq \theta_m \leq 1 - A_m \tau \quad (m=1, 2, \dots, M), \end{aligned}$$

$$\begin{aligned}
&\xi_1=0 \text{ and } 0 \leq \xi_m < 1 \quad (m=2, 3, \dots, M), \\
&\theta_m + w_m \tau - y_m \tau \geq 0 \quad (m=1, 2, \dots, M), \\
&\xi_m - \xi_\ell + \theta_m + A_m \tau \leq R z_{m\ell}, \\
&\xi_\ell + \theta_\ell + A_\ell \tau - \theta_m - 1 \leq R z_{m\ell}, \\
&\xi_\ell - \xi_m + \theta_\ell + A_\ell \tau \leq R(1 - z_{m\ell}), \\
&\xi_m + \theta_m + A_m \tau - \xi_\ell - 1 \leq R(1 - z_{m\ell}), \\
&\text{and } z_{m\ell} = 0 \text{ or } 1,
\end{aligned}$$

where  $\tau = 1/T$ ,  $\theta_m = G_m/T$  and  $\xi_m = x_m/T$ . Finally, the optimal signal setting based upon the criterion of minimum overall delay can be obtained by solving the following problem:

$$\begin{aligned}
(6.22) \quad &\min_{\underline{G}, \underline{x}, T} \{ D(\underline{G}, \underline{x}, T) \} \\
&\text{subject to (4.3), (6.16) through (6.18) and} \\
&0 < \rho_m(\underline{G}, \underline{x}, T) < 1 \quad (m=1, 2, \dots, M),
\end{aligned}$$

where  $\rho_m(\underline{G}, \underline{x}, T) = y_m T / g_m(\underline{G}, \underline{x}, T)$  and  $D(\underline{G}, \underline{x}, T)$  is defined by (6.12) and (6.13) with  $\rho_m(\underline{S}, T)$  replaced by  $\rho_m(\underline{G}, \underline{x}, T)$ .

The approximation strategy mentioned in Section 5 will supply a good approximation of optimal solutions of this problem. In particular, when only streams of the first priority are taken into consideration, the approximation strategy leads to the method that approximates Mixed Integer Nonlinear Programming Problem (6.22) by combination of Mixed Integer Programming Problem (6.21) with  $\tau$  fixed at  $1/T$  and a one dimensional

minimization problem similar to that in Step 4 mentioned above.

### 7. 7. Concluding remarks

In Sections 7. 2 and 7. 3, under rather practical assumptions, the criteria for undersaturation of the whole intersection are derived and mean effective green times of streams of low priorities are determined. Note that the sufficiency of the criterion remains valid in case of dependent arrivals, because as shown in Chapter 5, Condition (2. 2) is sufficient for a fixed-cycle traffic light queue with dependent arrivals to be undersaturated. The criterion may be important in design of intersections.

In Sections 7. 4 on, the optimal signal settings based upon the criterion of minimizing the degree of saturation and that of minimum overall delay are dealt with by use of results in Sections 7. 2 and 7. 3. Note that these optimal signal settings take streams both of the first priority and of lower priorities into account. It is shown that the criterion of minimizing the degree of saturation leads to the well-known rule of thumb and that this criterion is closely related to the criterion of overall minimum delay. The approximation algorithms for the optimal signal setting based upon the criterion of minimum overall delay are presented, which are combination of linear programming or mixed integer programming and one-dimensional minimization technique and are a refinement of Webster's (1958) method. Conditions for the approximation

algorithm to supply a good approximation are obtained in case of nonoverlapping phases. In case of overlapping phases, or intersections with complicated layout, these approximation algorithms will supply a good approximation in considerably short computation time. It is hoped to make practical trials of these approximation algorithms. Note that all results in Sections 7. 4 on remain valid with simple modification for optimal signal settings with the constraint  $G_n^0 \leq G_n \leq G_n^1$  ( $n=1,2,\dots,N$ ) or  $S_k^0 \leq S_k \leq S_k^1$  ( $k=1,2,\dots,K$ ) instead of the constraint  $G_n \geq 0$  ( $n=1,2,\dots,N$ ) or  $S_k \geq 0$  ( $k=1,2,\dots,K$ ), where  $G_n^0$  and  $G_n^1$  ( $S_k^0$  and  $S_k^1$ ) are the maximum and the minimum green times of the  $n$ th phase (the  $k$ th stage).

## Chapter 8. Conclusion

In the past decade, traffic problems such as air pollution, traffic congestion, traffic noise and traffic accidents have begun to give not only a technological but also sociological challenge in various countries in the world. The theory of traffic flow is one of the main approaches to the traffic problems. This thesis has been devoted to showing many fundamental results for several basic problems in the theory of traffic flow. As stated in Chapter 1, these problems are classified into the following main subjects: traffic flow, traffic queues and traffic control. Chapters 2 and 3 are concerned with "traffic flow". In Chapter 2, the low density inhomogeneous composed Poisson traffic flow on a road with an intersection is discussed and various transformations between distributions of time processes, those of space processes and those of observation processes are obtained. In Chapter 3, the most general low density inhomogeneous traffic flow is dealt with and it is shown that all time, space and observation processes can be expressed in terms of stochastic integrals of initial processes. This means that distributions of all processes are completely determined by the initial distributions. Moreover this applications to the low density homogeneous Poisson traffic flow lead to results

for the observation process which were obtained by Weiss and Herman (1962) and Rényi (1964) and to the well-known relation between time and space velocity distributions which was derived intuitively. It is to be noted that the transformations between initial distributions and distributions of time and space processes are of practical importance, because if initial distributions are known, then distributions of all time and space processes can be determined. It is easy to extend the results obtained in Chapters 2 and 3 to a traffic flow on a road network in an urban area. Throughout Chapters 2 and 3, the interactions between vehicles are assumed to be negligible. It is to be hoped that the present discussion will be extended to a medium density traffic flow in which the interactions between vehicles can not be disregarded.

Chapters 4, 5 and 6 are concerned with "traffic queues". In Chapter 4, it is shown that the fixed-cycle traffic light queue and the vehicle-actuated one reduce to the generalized model of the GI/G/1 queueing process, and necessary and sufficient conditions are derived under which their stationary distributions exist. Besides, the successive approximation method of the stationary distributions is presented. In Chapter 5 these results are extended to the traffic light queues



with dependent arrivals. Moreover, the results obtained in Chapters 4 and 5 are extended to the traffic light queues with departure headways depending upon positions in Chapter 6. It should be noted that this condition on departure headways is experimentally substantiated. In Chapter 6, the stationary queue length distribution of the fixed-cycled traffic light queue is also obtained. It is hoped that this result will be extended to a stationary distribution of an actual vehicle-actuated traffic light queue.

Chapter 7 is concerned with "traffic control". In the chapter, criteria for undersaturation of the whole intersection are derived and the optimal signal settings of the fixed-cycle traffic light based upon the criterion of minimizing the degree of saturation of the whole intersection and that of minimum overall delay are dealt with. It is shown that the former criterion leads to the well-known rule of thumb for optimal split and that it is closely related to the latter criterion. Moreover approximation algorithms for the optimal signal setting based upon the latter criterion are presented, which are a refinement of Webster's method. It is hoped to make practical trials of these approximation algorithms. Our future effort should be directed to develop efficient traffic control systems. For this purpose, results

obtained for stochastic controls of queueing systems  
(for example, Mine and Ohno (1971 a)) appear to be useful.

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### Appendix I

Consider a traffic flow composed of vehicles with velocity  $v_1$  and with velocity  $v_2$ . Let  $v_i \in K_i (i=1,2)$ , for disjoint sets  $\{K_i\}$ . Suppose that  $x(I_1)$  and  $x(I_2)$  are independent and  $E(x(I_i)) = \lambda_i > 0$ , for  $0 < a < 1$ ,  $\int_{K_1} dF(v, I_i) = a$ . Therefore, from property (E),

$$p(z; x(I_1, K_1)) = p(1 + a(z-1); x(I_1))$$

and

$$p(z; x(I_2, K_2)) = p(1 + (1 + (1-a)(z-1)); x(I_2)).$$

Let  $I = [0, 1)$ ,  $I_i = [-i, -i+1)$ ,  $v_i = i$  and  $t=1$ . It follows from (2.9) in Chapter 3 that

$$p(z; x_1(I, K_1)) = p(z; x(I_1, K_1))$$

and

$$p(z; x_1(I)) = p\left(z; \sum_{i=1}^2 x(I_i, K_i)\right) = \prod_{i=1}^2 p(z; x(I_i, K_i)).$$

By (D<sub>2</sub>),  $\int_{K_1} dF_1(v, I) = a\lambda_1 / \{a(\lambda_1 - \lambda_2) + \lambda_2\}$ . Hence,  $F_1(v, I)$  cannot satisfy property (E).

## APPENDIX II

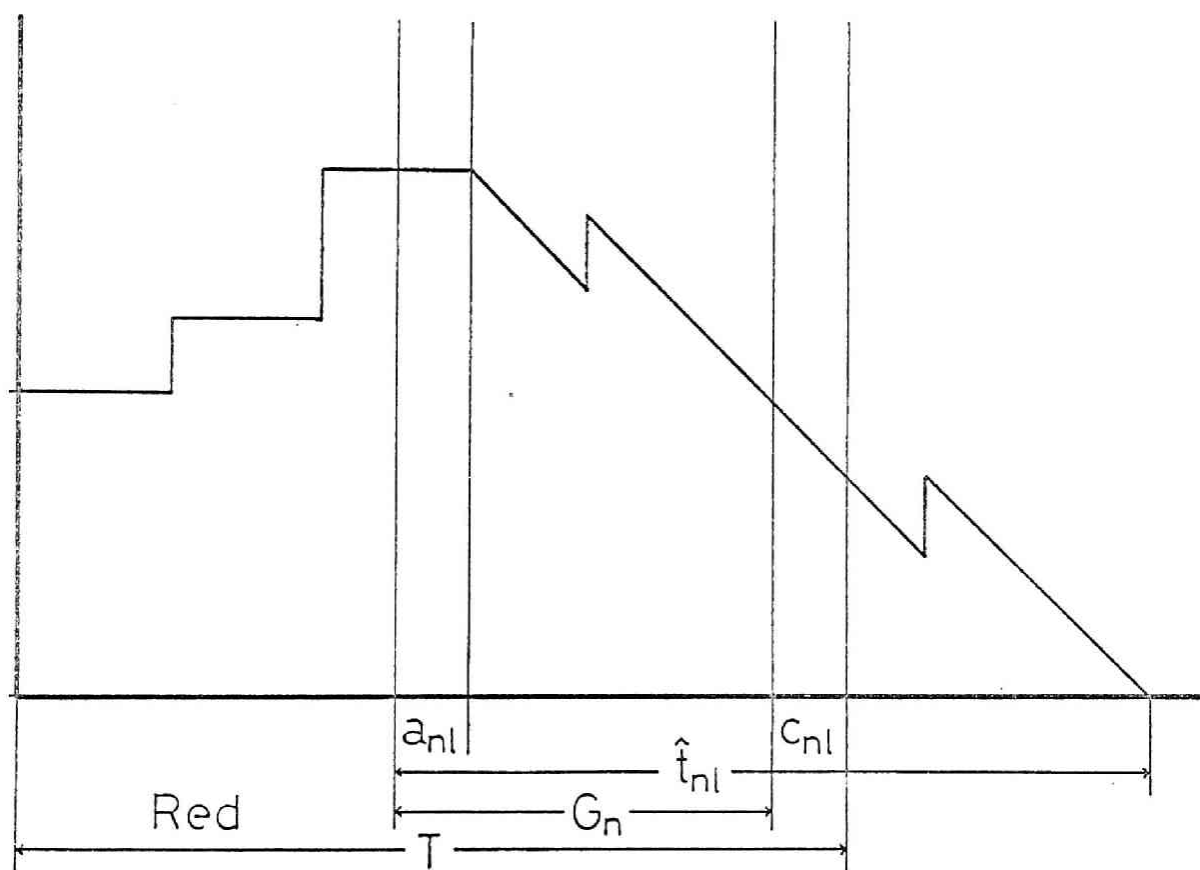
The block of the  $(n,m)$  stream consists of the maximum of the time intervals during which the queues of the first priority at the beginning of the green period discharge and the time intervals blocked by vehicles of the first priority which pass through the intersection without delay. Denote by  $b_{nm}^{(1)}$  and  $b_{nm}^{(2)}$  the mean lengths of the maximum discharge time and of the sum of blocked time intervals, respectively. Clearly,

$$(A. 1) \quad b_{nm} = b_{nm}^{(1)} + b_{nm}^{(2)}.$$

Let the  $(n,l)$  stream be one of the streams of the first priority which obstruct the  $(n,m)$  stream and denote by  $t_{nl}$  its discharge time in stationary states which is measured at its stop line and by  $\underline{E}(\cdot)$  the expectation of  $\cdot$  with respect to the stationary probability  $\underline{P}$ . From (2. 2) and (3. 1), this means that

$$(A. 2) \quad y_{nl} T < G_n + c_{nl} - a_{nl}.$$

In order to find  $\underline{E}(t_{nl})$ , consider the modified process which represents the total remaining departure headway of the  $(n,l)$  queue under the situation that the signal would remain indefinitely green after the beginning of the green period. Denote by  $\hat{t}_{nl}$  the discharge time for this modified process (see, Fig. A). Then



Modified process

FIG. A

$$(A. 3) \quad \underline{E}(\hat{t}_{nl}) = a_{nl} + y_{nl}(T - G_n - c_{nl} + \underline{E}(\hat{t}_{nl})) + q_{nl},$$

where  $q_{nl}$  is the stationary mean value of the total remaining departure headway at the beginning of the cycle. Consequently,

$$(A. 4) \quad \underline{E}(\hat{t}_{nl}) = \{y_{nl}(T - G_n - c_{nl}) + a_{nl} + q_{nl}\} / (1 - y_{nl}).$$

Since  $t_{nl} \leq G_n + c_{nl}$  almost surely (a.s.) and if  $x < G_n + c_{nl}$ , then  $\underline{P}\{t_{nl} \leq x\} = \underline{P}\{\hat{t}_{nl} \leq x\}$ ,

$$(A. 5) \quad \underline{E}(t_{nl}) = \underline{E}(\hat{t}_{nl}) - \int_{G_n + c_{nl}}^{\infty} (x - G_n - c_{nl}) d\underline{P}\{\hat{t}_{nl} \leq x\}.$$

On the other hand, in the same way as in (A. 3),

$$\begin{aligned} \int_{G_n + c_{nl}}^{\infty} (x - G_n - c_{nl}) d\underline{P}\{\hat{t}_{nl} \leq x\} &= q_{nl} + y_{nl} \int_{G_n + c_{nl}}^{\infty} (x \\ &\quad - G_n - c_{nl}) d\underline{P}\{\hat{t}_{nl} \leq x\}. \end{aligned}$$

This equation yields

$$(A. 6) \quad \int_{G_n + c_{nl}}^{\infty} (x - G_n - c_{nl}) d\underline{P}\{\hat{t}_{nl} \leq x\} = q_{nl} / (1 - y_{nl}).$$

The combination of Equations (A. 4) through (A. 6) leads to

$$(A. 7) \quad \underline{E}(t_{nl}) = \{y_{nl}(T - G_n - c_{nl}) + a_{nl}\} / (1 - y_{nl}).$$

Since  $\underline{E}(t_{nl})$  is equal to the mean blocked time of the  $(n, m)$  stream by the saturation flow of the  $(n, l)$  stream,  $b_{nm}^{(1)}$  is given by

$$(A. 8) \quad b_{nm}^{(1)} = \max_{\ell} \{E(t_{n\ell})\},$$

where max is taken over all streams  $(n, \ell)$  of the first priority that obstruct the  $(n, m)$  stream. It is to be noted that, precisely speaking,  $b_{nm}^{(1)} = E\{\max_{\ell} t_{n\ell}\}$  but this exact expression is too complicated to be used in the analysis of the optimal signal settings. Since the differences between the mean starting delays or the mean clearance times of streams of the first priority are usually negligible, it is legitimate in most cases to set the stream which attains  $b_{nm}^{(1)}$  as the stream which attains the maximum  $y$ -value. Put

$$(A. 9) \quad \hat{y}_{nm} = \max_{\ell} \{y_{n\ell}\}$$

and denote by  $\hat{a}_{nm}$  and  $\hat{c}_{nm}$  the mean starting delay and the mean clearance time of the stream which attains  $\hat{y}_{nm}$ . Therefore,

$$(A. 10) \quad b_{nm}^{(1)} = \{\hat{y}_{nm}(T - G_n - \hat{c}_{nm}) + \hat{a}_{nm}\} / (1 - \hat{y}_{nm}).$$

Note that  $\hat{y}_{nm}$ ,  $\hat{a}_{nm}$  and  $\hat{c}_{nm}$  are constant even if  $T$  or  $G_n$  varies.

In order to find  $b_{nm}^{(2)}$ , suppose that the  $(n, m)$  vehicle at the head of the queue can not move during the time interval of constant length  $\mathcal{L}_{nm}$  immediately before any vehicle of the first priority has passed the



conflict point with the  $(n,m)$  stream. Following Oliver (1962), time intervals longer than  $d_{nm}$  between successive arrivals of vehicles of the first priority are called gaps. Thus the  $(n,m)$  vehicle at the head of the queue can not move until the gap appears. This kind of problem has been discussed as the highway merging problem by many authors. Put

$$(A. 11) \quad \hat{\lambda}_{nm} = \sum_l \lambda_{nl},$$

where the sum is over all streams  $(n,l)$  of the first priority that obstruct the  $(n,m)$  stream. According to Oliver (1962 a), Tanner (1962), Weiss and Maradudin (1962) and others, the mean intergap headway is given by

$$(A. 12) \quad \exp(\hat{\lambda}_{nm} d_{nm}) / \hat{\lambda}_{nm}.$$

Since one intergap headway is composed of one gap and one blocked time interval, the mean length of the blocked interval in each intergap headway is given by

$$(A. 13) \quad \exp(\hat{\lambda}_{nm} d_{nm}) / \hat{\lambda}_{nm} - 1 / \hat{\lambda}_{nm}.$$

Thus, by (A. 12) and (A. 13),

$$(A. 14) \quad b_{nm}^{(2)} = \{1 - \exp(-\hat{\lambda}_{nm} d_{nm})\} (G_n + c_{nm} - b_{nm}^{(1)}).$$

Note that this equation is correct only when the arrivals of the vehicles of the first priority are

Poisson processes. When the arrivals are generalized Poisson processes, as is the present case,  $\lambda_{n\ell}$  in (A. 11) should be replaced theoretically by  $\lambda_{n\ell}$  divided by the mean member of  $(n, \ell)$  vehicles which arrive simultaneously. In this paper, however,  $\hat{\lambda}_{nm}$  is defined by (A. 11), because the arrival of the group of vehicles raises actually the blocked time interval of rather longer length than the arrival of a single vehicle and (A. 10) is underestimated. Thus, the combination of (A. 1), (A. 10) and (A. 14) yields

$$(A. 15) \quad b_{nm} = (G_n + \hat{c}_{nm}) \{ 1 - \exp(-\hat{\lambda}_{nm} \alpha_{nm}) / (1 - \hat{y}_{nm}) \} \\ + \exp(-\hat{\lambda}_{nm} \alpha_{nm}) (\hat{y}_{nm} T + \hat{a}_{nm}) / (1 - \hat{y}_{nm}).$$

In the above, it is assumed that  $\hat{a}_{nm} + \hat{\mu}_{nm} \leq \alpha_{nm}$ ; otherwise, the difference between right-hand and left-hand terms is negligible.

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